

Comment on: A note on the discontinuity problem in Heston's stochastic volatility model

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ABSTRACT

Guo and Hung [2007] recently studied the complex logarithm present in the characteristic function of Heston's stochastic volatility model. They proposed an algorithm for the evaluation of the characteristic function which is claimed to preserve its continuity. We show their algorithm is correct, although their proof is not.

Keywords: Complex logarithm, stochastic volatility, Heston, characteristic function.

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JEL Classification: C63, G13.

Under the risk-neutral pricing measure the Heston [1993] stochastic volatility model is specified by the following set of stochastic differential equations:

$$\begin{aligned}dS(t) &= \mu(t)S(t)dt + \sqrt{v(t)}S(t)dW_S(t) \\dv(t) &= -\kappa(v(t) - \theta)dt + \omega\sqrt{v(t)}dW_v(t)\end{aligned}\tag{1}$$

where the Brownian motions satisfy $dW_S(t) \cdot dW_v(t) = \rho dt$. The underlying asset S has a stochastic variance v , which is modelled as a mean-reverting square root process. The parameter κ is the rate of mean reversion of the variance, θ is the long-term level of variance and ω is the volatility of variance. Finally, the drift $\mu(t)$ is used to fit to the forward curve of the underlying.

Heston solved the (extended) characteristic function as:

$$\phi(u) = \mathbb{E}\left[e^{iu \ln S(T)}\right] = \exp(iuf + A(u, \tau) + B_v(u, \tau) \cdot v(0))\tag{2}$$

where $u \in \mathbb{C}$, f is shorthand for $\ln F(T)$, the logarithm of the forward price, and τ denotes the time to maturity. Both functions A and B_v follow from the usual Ricatti equations, and it is A which

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contains a complex logarithm. The solution for B_v and ordinary differential equation for A are:

$$B_v(u, \tau) = \frac{\beta(u) - D(u)}{\omega^2} \frac{1 - e^{-D(u)\tau}}{1 - G(u)e^{-D(u)\tau}} \quad \frac{dA(u, \tau)}{d\tau} = \kappa\theta B_v(u, \tau) \quad (3)$$

where:

$$\begin{aligned} \hat{\alpha}(u) &= -\frac{1}{2}u(i + u) & \beta(u) &= \kappa - \rho\omega u i & \gamma &= \frac{1}{2}\omega^2 \\ D(u) &= \sqrt{\beta(u)^2 - 4\hat{\alpha}(u)\gamma} & c(u) &= \frac{\beta(u) + D(u)}{\beta(u) - D(u)} & G(u) &= 1/c(u) \end{aligned} \quad (4)$$

A frequently cited solution to the ODE in (3) is:

$$A(u, \tau) = \kappa\theta\omega^{-2}((\beta(u) + D(u))\tau - 2 \ln \psi_1(u, \tau)) \quad \psi_1(u, \tau) = \frac{c(u)e^{D(u)\tau} - 1}{c(u) - 1} \quad (5)$$

Equation (5) is referred to as Formulation 1 in Lord and Kahl [2006, 2007]. It is well-known that this formulation causes problems if one restricts the complex logarithm to its principal branch, as pointed out in e.g. Schöbel and Zhu [1999] and Kahl and Jäckel [2005]. The reason for this is that $\psi_1(u, \tau)$, as a function of the real part of u , can and for some parameter configurations does cross the negative real line, leading to a phase jump of its complex logarithm.

One might argue that this is much ado about nothing, as an obvious way to avoid such problems is to integrate the defining ODE for A directly, as has indeed been mentioned in Lord and Kahl [2006, 2007] and Shaw [2006]. By construction this will lead to the correct and continuous solution. Practitioners however prefer closed-form solutions if they are available, for the simple reason of computational speed. By numerically integrating the ODE, one would forsake the comparative advantage of the Heston model over other, more complicated models, as the option price is then represented as a double integral instead of a single one.

Returning to (5), Guo and Hung [2007] recognised the problems associated with this solution, and argued that in the following formulation the logarithm can be restricted to its principal branch:

$$A(u, \tau) = \kappa\theta\omega^{-2}((\beta(u) - D(u))\tau - 2 \ln \psi_2(u, \tau)), \quad \psi_2(u, \tau) = \frac{1 - c(u)e^{D(u)\tau}}{1 - c(u)} e^{-D(u)\tau} \quad (6)$$

Equation (6) is equivalent to Formulation 2 in Lord and Kahl [2006, 2007], and first appears in the literature in Bakshi, Cao and Chen [1997, eq. A.11]. Lord and Kahl [2006] and Gatheral [2006] conjectured that evaluating function A as in (6) leads to no complex discontinuities. This conjecture was finally proved without any restrictions on parameters or on u in Lord and Kahl [2007]². While the finding of Guo and Hung is therefore correct, the argument they provide for using this formulation is not. They argue that:

$$\ln \psi_2(u, \tau) = \ln(c(u)e^{D(u)\tau} - 1) - \ln((c(u) - 1)e^{D(u)\tau}) \quad (7)$$

² An initial proof under parameter restrictions was given in Lord and Kahl [2006]. Fahrner [2007] considered the case where $\text{Im}(u) = -1/2$, also under the same parameter restrictions. Finally, the proof of Albrecher, Mayer, Schoutens and Tistaert [2007] requires the restriction that $\text{Im}(u) \leq -1$.

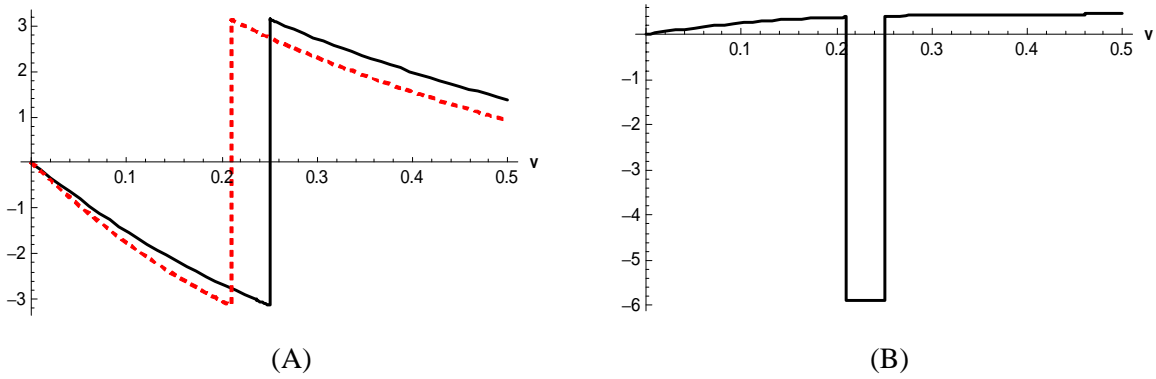


Figure 1: Complex discontinuities in the Heston model

Parameters: $\kappa = 2$, $\omega = 1$, $\rho = -0.3$, $\theta = 0.09$, $v(0) = 0.09$, $S(0) = 100$, $\mu = 0.05$, $\tau = 5$

(A) $\text{Im} \log[c(u)\exp(d(u)\tau) - 1]$ (black) vs. $\text{Im} \log[(c(u) - 1)\exp(d(u)\tau)]$ (red, dashed) with $u = v - 3.5i$

(B) The imaginary part of equation (6), i.e. the difference of the two lines in (A)

and that the logarithm of $c(u)e^{D(u)\tau} - 1$ has the same rotation count number as $(c(u) - 1)e^{D(u)\tau}$. Therefore, any phase jumps will be cancelled out, so that we can restrict both complex logarithms to their principal branch. They conclude that the function A should be evaluated as in (5).

This is wrong on two accounts. First of all, $c(u)e^{D(u)\tau} - 1$ does not have the same rotation count number as $(c(u) - 1)e^{D(u)\tau}$, as is evident from Figure 1a. Evaluating the logarithm of $\psi_2(u, \tau)$ as in (6), by restricting both logarithms to their principal branch will therefore lead to a discontinuous function, see Figure 1b. Second, $\psi_2(u, \tau)$ never crosses the negative real line, as proven in Lord and Kahl [2007]. Therefore, evaluating A as in (5) constitutes a continuous function, whereas evaluating A as in (6) does not. The parameters in Figure 1 are those used in the second example of Broadie and Kaya [2006, Table 2]. Put otherwise - Formulation 2 will produce correct option prices, whereas following the algorithm in (6) leads to incorrect ones.

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