

Fourier inversion methods in finance

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Abstract

In finance a variety of models allow for analytical or semi-analytical solutions of their characteristic functions despite having no closed form expressions of their respective distributions. Models with these characteristics include affine-jump diffusions, regime switching and Lévy processes. By aid of the Plancharel/Parseval identity pricing European options thus reduces to a quadrature problem over an infinity domain. Here we discuss different numerical schemes, such as FFT, saddlepoint approximation and adaptive quadrature schemes to solve this problem efficiently. In addition, we analyse the pricing of early exercise options, such as Bermudan and American options, by means of inverse Fourier transformation and present the CONV algorithm.

1 Introduction

The seminal work of Heston [Hes93a], in the context of a stochastic volatility model, is the first appearance of Fourier inversion in option pricing where results were obtained which cannot be derived by other means. Ever since then Fourier inversion has become more and more popular. Prior to Heston, Stein and Stein [SS91] had already utilised Fourier inversion techniques to derive the stock price distribution in their stochastic volatility model which we nowadays would refer to as Schöbel-Zhu model with uncorrelated stochastic volatility. Before analysing the pricing of European options by means of Fourier inversion in greater depth we recall the pricing of a European call option

$$\begin{aligned}
 \frac{C(t)}{P(t, T)} &= \mathbf{E}_t^{\mathbb{Q}_T} [(S_T - K)^+] & (1) \\
 &= \mathbf{E}_t^{\mathbb{Q}_T} [S_T \cdot 1_{\{S_T > K\}}] - K \cdot \mathbf{E}_t^{\mathbb{Q}_T} [1_{\{S_T > K\}}] \\
 &= \frac{S_t}{P(t, T)} \cdot \mathbf{E}_t^{\mathbb{S}_T} [1_{\{S_T > K\}}] - K \cdot \mathbb{P}(S_T > K) \\
 &= F(t, T) \cdot \mathbb{S}_T(S_T > K) - K \cdot \mathbb{P}(S_T > K), & (2)
 \end{aligned}$$

where $P(t, T)$ is the time t value of a zero coupon bond expiring in T corresponding to the T forward measure \mathbb{Q}_T and $F(t, T)$ refers to the T forward of the underlying asset as seen at time t . \mathbb{S}_T denotes the stock price measure, where $S(\cdot)$ is the numeraire. Note that (2) has essentially the form of the celebrated Black-Scholes formula. Both cumulative probabilities can be found by inverting the forward characteristic function:

$$\mathbb{P}(S_T > K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[e^{iuk} \frac{\varphi(u)}{iu} \right] du \quad (3)$$

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$$\mathbb{S}(S_T > K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[e^{iuk} \frac{\varphi(u)}{iu \cdot \varphi(-i)} \right] du \quad (4)$$

where k is the logarithm of the strike price K . Formulations (3) and (4) dates back to Lévy [Lév25] although his approach is restricted to cases where the random variable is strictly positive. Gurland [Gur48] and Gil-Pelaez [GP51] derived this more general inversion theorem and we refer the interested reader to Lukacs [Luk70] for a detailed account of the history of these approaches.

1.1 Plancharel/Parseval identity

At the very heart of Fourier inversion methods is the Plancharel or Parseval identity [Tit75], which essentially states that the product of two functions $f, g \in L^1$ can be directly expressed as the product of their respective Fourier transforms, i.e.

$$\int_{-\infty}^{\infty} f(x) \cdot g(x) dx = \frac{1}{2\pi} \int_{-\infty+i\alpha}^{\infty+i\alpha} \hat{f}(z) \cdot \hat{g}(-z) dz \quad (5)$$

where α is the contour of integration. The left hand side of this equation can be seen as the pricing of a contingent claim in some risk neutral measure, where f refers to the density of the underlying asset and g represents the respective payoff function, i.e. in the T -forward measure \mathbb{Q}_T , we have

$$\frac{V(t)}{P(t, T)} = \mathbb{E}_t^{\mathbb{Q}_T} [g(S_T)] = \int_{-\infty}^{\infty} f(x) \cdot g(x) dx . \quad (6)$$

Using a Fourier transform becomes attractive if there exists no closed form solution for the density of the underlying asset f , or if this density involves further computational intensive operations as in the case of (most) Lévy processes, which we cover in section 2.1.

Applying equation (5) also requires to compute the Fourier inversion of the payoff function g . In case of a European call option with payoff function

$$g(x) = (e^x - e^k)^+ \quad (7)$$

we can directly compute

$$\hat{g}(z) = \int_{\mathbb{R}} e^{izx} (e^x - e^k)^+ dx = -\frac{e^{k(iz+1)}}{z(z-i)} . \quad (8)$$

Here we recognize that the Fourier transform has two singularities at $z = 0$ and $z = i$. Thus if we move the integration contour α in (5) over those singularities we have to apply residual calculus. As such the Fourier inversion problem for a European call options reads

$$C(S, K, \tau, \alpha) = R(S, K, \alpha) + \frac{1}{2\pi} \int_{-\infty-i\alpha}^{\infty-i\alpha} e^{-izk} \frac{\varphi(z-i)}{-z(z-i)} dz , \quad (9)$$

where the residue term equals

$$R(S, K, \alpha(k)) = F \cdot 1_{\{\alpha \leq 0\}} - K \cdot 1_{\{\alpha \leq -1\}} - \frac{1}{2} (F \cdot 1_{\{\alpha=0\}} - K \cdot 1_{\{\alpha=-1\}}) . \quad (10)$$

It is worth mentioning that this formula is now equivalent to the one derived by Carr and Madan [CM99]. The idea of Carr and Madan was to consider a dampened call price and to compute its Fourier inversion which reads

$$C(S, K, \tau, \alpha) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \frac{e^{-ivk} \varphi(v - i(\alpha + 1))}{-(v - i\alpha)(v - i(\alpha + 1))} dv = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi_\alpha(v) dv . \quad (11)$$

The additional free parameter α is essentially the integration contour used in the Parseval identity (9) as pointed out by Lewis [Lew01]. Although this approach was new to the area of option pricing, the idea of damping functions on the positive real line in order to be able to find their Fourier transform is an idea that goes back to at least Dubner and Abate [CM99]. Note that for most payoff functions g used in real financial applications, the Fourier transformation is available in closed form.

1.2 Pricing early exercise options

The best known examples of options with early exercise features are American and Bermudan options. American options can be exercised at any time prior to the option's expiry, whereas Bermudan options can only be exercised at prespecified future dates. In order to facilitate the exposure, we will focus solely on Bermudan options. American options can always be approximated by Bermudans with a large number of exercise opportunities, or by using extrapolation techniques such as in [CCS02].

Now we have demonstrated in the previous section that the Plancharel or Parseval identity lies at the basis of pricing European options via Fourier inversion methods, the next question is whether we can price options with early-exercise features in a similar framework. Before we address this question, let us first introduce some notation. Let us define the set of exercise dates as $\mathcal{T} = t_1, \dots, t_M$ and $0 = t_0 \leq t_1$. Finally, we define $\Delta_m = t_{m+1} - t_m$. The exercise payoff one obtains when the option holder exercises equals $E(t, S(t))$. Using this notation, the Bermudan option price can then be found via backward induction as:

$$\begin{cases} V(t_M, S(t_M)) = E(t_M, S(t_M)) \\ C(t_m, S(t_m)) = N(t_m) \cdot \mathbb{E}_{t_m}^{\mathcal{N}} [V(t_{m+1}, S(t_{m+1})) \cdot N(t_{m+1})^{-1}] \\ V(t_m, S(t_m)) = \max \{C(t_m, S(t_m)), E(t_m, S(t_m))\} \\ V(t_0, S(t_0)) = C(t_0, S(t_0)) \end{cases} \quad (12)$$

for $m = M - 1, \dots, 1$ with pricing measure \mathcal{N} associated to numeraire N . Here C denotes the continuation value of the option (not to be confused with the call option in equations (9) and (11)). Finally, V denotes the value of the option immediately prior to the exercise opportunity.

The continuation values in (12) are all applications of the risk-neutral valuation formula, so that if we can value the continuation value via Fourier inversion methods, we can certainly value Bermudan options using Fourier inversion methods. The key difference is that whereas the value V of the option, and its Fourier transform, are typically known analytically for the most common types of European contracts, the corresponding value functions for Bermudan options are typically only known numerically.

As far as we are aware, the first paper to consider pricing Bermudan options using the Fourier inversion techniques of [CM99] was [O'S05], who extended the QUAD method of [AWDN03] to allow for models where the density is not known in closed-form. Picking up on work by Reiner [Rei01], Lord et al. [LFBO08] simplified the complexity of the QUAD method and O'Sullivan's algorithm from $\mathcal{O}(MN^2)$ to $\mathcal{O}(MN \log N)$, M being the number of exercise dates, and N being the number of node points used to discretise the underlying asset, by recognising that the continuation value at t_{m-1} in (12) can be written as a convolution of the value at t_m and the transition density. The so-called CONV method of Lord et al. [LFBO08] utilises the *Fast Fourier Transform* (FFT) (see also section 3.1) to approximate the convolutions. Finally, we mention recent work of Fang and Oosterlee [FO08b], in which Bermudan options are priced via Fourier-cosine series expansions. This method is of the same complexity as the CONV method, though a large advantage is that the exercise boundary is solved directly, as in the QUAD method. Instead of being approximated, the cosine series coefficients can therefore be calculated exactly, which improves the convergence of the algorithm. Whereas the convergence of the CONV method, as we will see later in section 3.4 is typically dictated by the chosen Newton-Cotes rule for integration, the convergence of the COS method is dictated by the rate of decay of the characteristic function only. For

characteristic functions that decay faster than a polynomial (which is the case for e.g. the Black-Scholes model), this certainly can be advantageous.

It has been shown in [LFB08] that for Bermudan options the CONV algorithm is competitive with partial-integro differential (PIDE) methods. PIDE methods are however advantageous for valuing American options, as well as with respect to grid choice, which in the CONV method is restricted due to the use of the FFT for the calculation of the convolutions. We will provide more details on the CONV and COS methods in section 3.4.

2 Characteristic function

In this section, we discuss various models used in different areas of quantitative finance which allow for a closed form solution of its characteristic function. Knowing the characteristic function is particularly useful if the distribution of the underlying asset is not known in closed form or if it is computationally expensive to calculate.

2.1 Lévy processes

First we give a brief introduction about Lévy processes. For more background information we refer the interested reader to Cont and Tankov [CT04] for an extensive manuscript on the usage of Lévy processes in a financial context and to Sato [Sat99] for a detailed analysis of Lévy processes in general. Papapantoleon [Pap06] provides a good short introduction to the applications of Lévy processes in mathematical finance.

A Lévy process, is a continuous-time stochastic process with stationary independent increments. Its most well-known examples are Wiener processes or Brownian motions, and Poisson processes. To be precise, a càdlàg, adapted, real-valued process $L(t)$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $L(0) = 0$, is a Lévy process if:

1. it has independent increments
2. it has stationary increments
3. it is stochastically continuous, i.e. for any $t = 0$ and $\epsilon > 0$ we have

$$\lim_{s \rightarrow t} \mathbb{P}(|L(t) - L(s)| > \epsilon) = 0 . \quad (13)$$

Each Lévy process can be characterised by a triplet (σ, μ, ν) , representing the drift, diffusion and the jump distribution of the process. We require the measure ν to satisfy $\nu(0) = 0$ and

$$\int_{\mathbb{R}} \min(1, |x|) \nu(dx) < \infty . \quad (14)$$

The Lévy measure ν carries useful information about the path properties of the process. For example, if $\nu(\mathbb{R}) < \infty$, then almost all paths have a finite number of jumps on every compact interval. In this case the process is said to have finite activity. This is the case for e.g. compound Poisson processes. In contrast, if $\nu(\mathbb{R}) = \infty$, the process is said to have infinite activity. In terms of the Lévy triplet the characteristic function of the Lévy process for $u \in \mathbb{R}$ equals:

$$\varphi(u) = \mathbb{E} [e^{i \cdot u \cdot L_t}] = \exp \left(t \left(i\mu u - \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux \cdot 1_{\{|x| < 1\}}) \nu(dx) \right) \right) , \quad (15)$$

the Lévy-Khinchin formula. The exponent in (15) is referred to as the Lévy or characteristic exponent.

2.1.1 Variance Gamma model

In the Variance Gamma model [MCC98] the stock price process is modelled as:

$$S(t) = F(t) \cdot \exp(\omega \cdot t + \theta \cdot G(t) + \sigma \cdot W(G(t))) \quad (16)$$

where $W(t)$ is a standard Brownian motion, and $G(t)$ is a Gamma process with parameter ν . The forward price of the underlying asset at time t is denoted by $F(t)$. Finally, the parameter ω ensures that the exponential term has a mean equal to 1:

$$\omega = \frac{1}{\nu} \log \left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu \right) \quad (17)$$

As before we let $f(t) = \log F(t)$ and introduce $\tilde{f}(t) = f(t) + \omega t$. The conditional characteristic function is then specified as:

$$\varphi_{\text{VG}}(u) = \frac{\exp\left(iu\tilde{f}(T)\right)}{\left(1 - iu\left(\theta + \frac{1}{2}i\sigma^2u\right)\nu\right)^{\tau/\nu}}. \quad (18)$$

For further discussion on pricing options in the Variance Gamma model we refer to [Jäc09].

2.2 Affine (jump) diffusion models

Affine diffusion models allow for (semi)-analytical solutions of their respective characteristic functions, due to the fact that the partial differential equation associated with the risk neutral pricing formula (1) does, by the theorem of Feynman-Kac, translate into a system of Riccati equations in Fourier space using an appropriate Ansatz function. We first state the general result about affine diffusion models, and then analyse its most famous members, the models of Heston [Hes93a] and Schöbel-Zhu [SZ99], as well as their Hull-White [HW90] extension to include stochastic interest rates. Moreover the generality of this derivation does also cover the models which allow for joint jumps in the underlying and the volatility such as the ones proposed by Matytsin [Mat99] and [DPS00].

Deriving the characteristic function for an affine diffusion model such as Heston we can follow the lead by Duffie, Pan and Singleton [DPS00] and Duffie, Filipović and Schachermeyer [DFS03] and consider the following general diffusion model

$$d\mathbf{X}_t = \boldsymbol{\mu}(\mathbf{X}_t) dt + \boldsymbol{\sigma}(\mathbf{X}_t) d\mathbf{W}_t + d\mathbf{Z}_t, \quad (19)$$

with $\mathbf{X} = (X_1, \dots, X_n)$ denoting an n -dimensional stochastic process, $\boldsymbol{\mu} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $\boldsymbol{\sigma} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{n \times n})$ where \mathbf{W} refers to a standard Brownian motion in \mathbb{R}^n . The process \mathbf{Z} is a Lévy process with some strictly positive intensity $\boldsymbol{\lambda}(\mathbf{X}(t))$ and associated Lévy measure ν . Further we consider the characteristic function

$$\varphi(\mathbf{u}, x, t) := \mathbb{E} \left[e^{i\mathbf{u} \cdot \mathbf{X}_t} \right], \quad (20)$$

with $\mathbf{u} = (u_1, \dots, u_n)$. Under certain conditions on the drift $\boldsymbol{\mu}$, the diffusion $\boldsymbol{\sigma}$ and the intensity λ , the characteristic function is given as a solution of a system of Riccati equations. Using the Feynman-Kac theorem [Øks03, CT04], we know that equations (19) and (20) are equivalent to solving

$$\begin{aligned} \frac{\partial \varphi(u, x, t)}{\partial t} + \boldsymbol{\mu}(x) \frac{\partial \varphi(u, x, t)}{\partial x} + \frac{1}{2} \text{tr} \left[\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}(x)^T \frac{\partial^2 \varphi(u, x, t)}{\partial x^2} \right] \\ + \boldsymbol{\lambda}(x) \int_{\mathbb{R}^n} \left(\varphi(u, x + z, t) - \varphi(u, x, t) - z^T \cdot \mathbf{1}_{\{|z| < 1\}} \frac{\partial \varphi(u, x, t)}{\partial x} \right) \nu(dz) = 0, \end{aligned} \quad (21)$$

where $\text{tr}(A)$ is the trace of a matrix A and $\frac{\partial \varphi(u, x, t)}{\partial x} = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \dots, \frac{\partial \varphi}{\partial x_n} \right)$ is the vector of the partial derivatives. The boundary condition of (21) is given as

$$\varphi(\mathbf{u}, x, 0) = e^{i \cdot \mathbf{u} \cdot x}, \quad (22)$$

where x is the initial value of \mathbf{X} . We refer to an affine (jump) diffusion if the stochastic process (19) satisfies the following conditions

$$\boldsymbol{\mu}(x) = K_0 + K_1^T \cdot x, \quad \text{for } K_0 \in \mathbb{R}^n \text{ and } K_1 \in \mathbb{R}^{n \times n} \quad (23)$$

and

$$(\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}^T(x)) = H_0 + H_1 \cdot x, \quad \text{for } H_0 \in \mathbb{R}^{n \times n} \text{ and } H_1 \in \mathbb{R}^{n \times n \times n}, \quad (24)$$

as well as

$$\boldsymbol{\lambda}(x) = \lambda_0 + \lambda_1^T \cdot x, \quad \text{for } \lambda_0 \in \mathbb{R}^n \text{ and } \lambda_1 \in \mathbb{R}^{n \times n}. \quad (25)$$

The key result of Duffie, Pan and Singleton [DPS00] is that the characteristic function of an affine (jump) diffusion model is given as the solution of a system of Riccati equations. Therefore we assume that the characteristic function φ is given as

$$\varphi(u, x, t) = e^{A(t, u) + \mathbf{B}(t, u)^T \cdot x}. \quad (26)$$

Inserting this into (21) under the assumptions (23), (24) and (25) and dividing by φ yields

$$\begin{aligned} \frac{\partial A}{\partial t} + \left(\frac{\partial \mathbf{B}}{\partial t} \right)^T \cdot x + \mathbf{B}^T \cdot \boldsymbol{\mu}(x) + \frac{1}{2} \text{tr} [\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}(x)^T \cdot \mathbf{B}^T \cdot \mathbf{B}] \\ + \boldsymbol{\lambda}(x) \int_{\mathbb{R}^n} \left(e^{\mathbf{B}^T \cdot z} - 1 - \mathbf{B}^T \cdot z \cdot 1_{\{|z| < 1\}} \right) \nu(dz) = 0. \end{aligned} \quad (27)$$

We further denote

$$\Psi(u) = \boldsymbol{\lambda}(x) \int_{\mathbb{R}^n} \left(e^{i \cdot u^T \cdot z} - 1 - i \cdot u^T \cdot z \cdot 1_{\{|z| < 1\}} \right) \nu(dz). \quad (28)$$

Now we can sort terms in equation (27) with respect to their order in x , and thus the characteristic function is given as the solution of the complex valued ordinary differential equations

$$\frac{\partial A(t)}{\partial t} = K_0^T \cdot \mathbf{B}(t) + \frac{1}{2} \mathbf{B}(t)^T \cdot H_0 \cdot \mathbf{B}(t) + \lambda_0 \cdot \Psi(-i \mathbf{B}), \quad (29)$$

$$\frac{\partial \mathbf{B}(t)}{\partial t} = K_1^T \cdot \mathbf{B}(t) + \frac{1}{2} \mathbf{B}(t)^T \cdot H_1 \cdot \mathbf{B}(t) + \lambda_1 \cdot \Psi(-i \mathbf{B}), \quad (30)$$

subject to the boundary conditions $\mathbf{B}(0) = i \cdot \mathbf{u}$ and $A(0) = 0$.

2.2.1 Heston

Here we derive the bivariate characteristic function, with respect to the underlying and the stochastic volatility component, for the Heston model. Note that this also directly covers the characteristic function with respect the underlying asset, by setting $u_2 = 0$, which is essentially sufficient for pricing European contingent claims. The Heston model is affine in $X_t = \log F_t$ and v_t ,

$$dX_t = -\frac{1}{2} v_t dt + \sqrt{v_t} dW_t, \quad (31)$$

$$dv_t = \kappa(\theta - v_t) dt + \omega \sqrt{v_t} dZ_t, \quad (32)$$

with correlated Brownian motions $d\langle W, Z \rangle_t = \rho dt$ and the auxiliary variables given by

$$K_0 = \begin{pmatrix} 0 \\ \kappa\theta \end{pmatrix}, \quad K_{(1,1)} = 0, \quad K_{(1,2)} = \begin{pmatrix} -\frac{1}{2} \\ -\kappa \end{pmatrix}, \quad (33)$$

as well as

$$H_0 = 0, \quad H_{(1,1)} = 0, \quad H_{(1,2)} = \begin{pmatrix} 1 & \rho\omega \\ \rho\omega & \omega^2 \end{pmatrix}. \quad (34)$$

Thus we have to solve the following system of ordinary differential equations

$$\frac{\partial B_1(t)}{\partial t} = 0, \quad (35)$$

$$\frac{\partial B_2(t)}{\partial t} = \hat{\alpha}(u) - \beta(u)B_2(t) + \gamma B_2(t)^2, \quad (36)$$

$$\frac{\partial A(t)}{\partial t} = \kappa\theta B_2(t) \quad (37)$$

subject to the boundary conditions $B_1(0) = i \cdot u_1$, $B_2(0) = i \cdot u_2$ and $A(0) = 0$. The auxiliary variables we introduced are

$$\hat{\alpha} = -\frac{1}{2}u_1(i + u_1), \quad \beta = \kappa - \rho\omega u_1 i \quad \text{and} \quad \gamma = \frac{1}{2}\omega^2. \quad (38)$$

The solution can be derived as

$$B_1(t, u) = i \cdot u_1, \quad (39)$$

$$B_2(t, u) = \frac{\beta + D(u)}{\omega^2} \left(\frac{e^{D(u)t} - 1}{c(u)e^{D(u)t} - 1} \right), \quad (40)$$

$$A(t, u) = \frac{\kappa\theta}{\omega^2} \left((\beta + D(u))t - 2 \log \left(\frac{c(u)e^{D(u)t} - 1}{c(u) - 1} \right) \right), \quad (41)$$

where

$$c(u) = \frac{\beta + D(u) - i \cdot u_2 \omega^2}{\beta - D(u) - i \cdot u_2 \omega^2}, \quad (42)$$

and

$$D(u) = \sqrt{\beta^2 - 4\hat{\alpha}\gamma}. \quad (43)$$

We remark that for $D(u)$ we use the common convention that the real part of the square root is nonnegative. As the characteristic function is even in D , this is no restriction. We further refer to the reciprocal of c as

$$G(u) = \frac{\beta - D(u) - i \cdot u_2 \omega^2}{\beta + D(u) - i \cdot u_2 \omega^2}. \quad (44)$$

The Riccati equation for B_2 is essential when analysing the problem of moment explosion of the characteristic function as first observed by Andersen and Piterbarg [AP07].

Numerical evaluation of the characteristic function in the Heston model, involves mainly two different kind of problems. The first is the oscillatory nature when computing option prices via inverse Fourier transform, a problem we are going to analyse in greater detail in section 3. The second problem is the evaluation of a complex logarithm in equation (41). In fact, for the univariate case, continuity can be preserved simply by rearranging terms in the representation of the characteristic function as

$$B_2(t) = \frac{\beta(u) - D(u)}{\omega^2} \left(\frac{1 - e^{-D(u)t}}{1 - G(u)e^{-D(u)t}} \right), \quad (45)$$

such that taking the principal branch of the complex logarithm guarantees continuity, as initially outlined and discussed by Lord and Kahl [LK06], subsequently in Albrecher et al. [AMST07], and finally proven

without any parameter restrictions by Lord and Kahl in [LK10]. The added benefit of this alternative representation is that it is numerically more stable. This formulation was first used by Bakshi, Cao and Chen [BCC97], and has subsequently also appeared in Duffie, Pan and Singleton [DPS00], Schoutens, Simons and Tistaert [SST04] and Gatheral [Gat06], among others. As these authors used this alternative formulation, in hindsight it is not surprising that they did not mention any numerical difficulties in finding option prices. Schöbel and Zhu [SZ99] were first to mention such numerical issues, followed by a.o. [MN03, Lee05, KJ05]. An alternative solution is to apply the so-called *rotation count algorithm*, as introduced by Kahl and Jäckel [KJ05]. As of yet there is however no proof that this algorithm is guaranteed to work for all parameter configurations, so that one is better off using the formulation in (45).

2.2.2 Schöbel-Zhu

The characteristic function of the Schöbel-Zhu model can be deduced in an analogous manner. Despite the fact that the original approach of Schöbel-Zhu used different arguments to derive the characteristic function, it turns out that the affine diffusion approach allows to find a representation which is similar to the characteristic function of the Heston model. Thus solving the problem of complex discontinuities in the Heston model also solves this problem for the Schöbel-Zhu model. The Schöbel-Zhu model is affine in $X_t = \log F_t$, v_t and $z_t = v_t^2$,

$$dX_t = -\frac{1}{2}z_t dt + \sqrt{z_t} dW_t, \quad (46)$$

$$dv_t = \kappa(\theta - v_t) dt + \omega dZ_t, \quad (47)$$

$$dz_t = (-2\kappa z_t + 2\kappa\theta v_t + \omega^2) dt + 2\omega\sqrt{z_t} dZ_t. \quad (48)$$

We refer the interested reader to Kahl [Kah07] and Lord [Lor08] for a more detailed discussion on the derivation of the characteristic function in the Schöbel-Zhu model.

2.2.3 Stochastic interest rate extension

In recent years several publications have appeared on how to extend an affine-diffusion stochastic volatility model to include stochastic interest rates, which is particular interesting for the modelling of long dated foreign-exchange options such as TARNs (Target redemption notes) and PRDCs (Power reverse dual coupons) since the volatility of the foreign-exchange rate is mainly driven by the volatility of the domestic and foreign interest rate component. For simplicity often a single factor Hull-White short rate model is chosen for the domestic and foreign interest rates

$$dr_t = \kappa(\theta - r_t) dt + \sigma_t dW_t^{\mathbb{Q}}, \quad (49)$$

with bond price (in the respective currency and in the respective money market measure)

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r(u) du} \right]. \quad (50)$$

One example of this is the stochastic interest rate extension of the Schöbel-Zhu model, the so-called Schöbel-Zhu-Hull-White (SZHW) model of Van Haastrecht, Lord, Pelsser and Schrage [HLPS09], where the short rate processes can be directly coupled with the underlying asset price and its stochastic volatility, whilst still remaining an affine diffusion

$$dF(t) = F(t)(r_d(t) - r_f(t)) dt + F(t) \cdot z(t) dW_F^{\mathbb{Q}_d}(t), \quad (51)$$

$$dr_d(t) = \kappa_d(\theta_d(t) - r_d(t)) dt + \sigma_d dW_{r_d}^{\mathbb{Q}_d}(t), \quad (52)$$

$$dr_f(t) = [\kappa_f (\theta_f(t) - r_f(t)) - \rho_{(f,F)} \cdot \sigma_F(t) \cdot \sigma_f] dt + \sigma_f dW_{r_f}^{\mathbb{Q}_d}(t), \quad (53)$$

$$dz(t) = \kappa (\theta - z(t)) dt + \omega dW_z^{\mathbb{Q}_d}(t). \quad (54)$$

Here \mathbb{Q}_d denotes the domestic money market measure. Note that the correlation structure is fully flexible. In the Heston model the issue is rather more complicated, as one has to be much more careful not to violate the restriction on affine diffusions when extending the model using Hull-White stochastic interest rates. To this end, Giese [Gie04] and Andreasen [And06] split the volatility of the underlying in two separate parts, where the stochastic volatility is uncorrelated to the stochastic interest rate component

$$dF(t) = F(t) (r_d(t) - r_f(t)) dt + F(t) \cdot \left(\sigma_F(t) dW_{(F,1)}^{\mathbb{Q}_d}(t) + \sqrt{z(t)} dW_{(F,2)}^{\mathbb{Q}_d}(t) \right), \quad (55)$$

$$dr_d(t) = \kappa_d (\theta_d(t) - r_d(t)) dt + \sigma_d dW_{r_d}^{\mathbb{Q}_d}(t), \quad (56)$$

$$dr_f(t) = [\kappa_f (\theta_f(t) - r_f(t)) - \rho_{(f,F)} \cdot \sigma_F(t) \cdot \sigma_f] dt + \sigma_f dW_{r_f}^{\mathbb{Q}_d}(t), \quad (57)$$

$$dz(t) = \kappa (\theta - z(t)) dt + \omega \cdot \sqrt{z(t)} dW_z^{\mathbb{Q}_d}(t), \quad (58)$$

with $W_{(F,2)}$ uncorrelated to both domestic and foreign interest rates W_{r_d} and W_{r_f} respectively. By decoupling the volatility of the underlying in this way, one is somewhat restricted with regards to the correlation structure one can achieve. On the upside, the model remains affine and therefore tractable.

2.3 Miscellanea

Here we present some additional results related to Fourier inversion option pricing in computational finance. First we show that piecewise constant model parameters can easily be handled in an affine (jump) diffusion framework as long as the bivariate characteristic function

$$\varphi(u_1, u_2, \tau, x(t), v(t)) = \mathbb{E} \left[e^{i \cdot (u_1 \cdot x(T) + u_2 \cdot v(T))} \mid \mathcal{F}_t \right] \quad (59)$$

is known analytically. In section 2.2.1 we presented the bivariate-characteristic function for the Heston model (see also [Zhy10]) having the form

$$\varphi(u_1, u_2, \tau, x(t), v(t)) = e^{A(u_1, u_2, \tau) + B_1(u_1, u_2, \tau) \cdot x(t) + B_2(u_1, u_2, \tau) \cdot v(t)} \quad (60)$$

with A , B_1 and B_2 given in equation (39), (40) and (41) respectively.

Piecewise constant model parameters

Pricing in a framework of piecewise constant model parameters essentially requires a recursive application of the tower law, see Mikhailov and Nögel [MN03]. Suppose we have piecewise constant model parameters, given a termstructure of time $t_0 < t_1 < \dots < t_{n-1} < t_n$, then we can calculate

$$\begin{aligned} \mathbb{E} \left[e^{iu_1 x(t_n)} \right] &= \mathbb{E} \left[\mathbb{E} \left[e^{iu_1 x(t_n)} \mid \mathcal{F}_{t_{n-1}} \right] \right] \\ &= \mathbb{E} \left[\varphi(u_1, 0, \Delta t_n, x(t_{n-1}), v(t_{n-1})) \right] \\ &= \mathbb{E} \left[e^{A(u_1, 0, \Delta t_n) + B(u_1, 0, \Delta t_n) \cdot v(t_{n-1}) + i \cdot u_1 x(t_{n-1})} \right] \\ &= e^{A(u_1, 0, \Delta t_n)} \cdot \mathbb{E} \left[e^{i \cdot (-i \cdot B(u_1, 0, \Delta t_n) \cdot v(t_{n-1}) + u_1 x(t_{n-1}))} \right] \\ &= e^{A(u_1, 0, \Delta t_n)} \cdot \mathbb{E} \left[\varphi(u_1, u_2(t_{n-1}), \Delta t_{n-1}, x(t_{n-2}), v(t_{n-2})) \right] \\ &= e^{A(u_1, 0, \Delta t_n)} \cdot \mathbb{E} \left[e^{A(u_1, u_2(t_{n-1}), \Delta t_{n-1}) + B(u_1, u_2(t_{n-1}), \Delta t_{n-1}) \cdot v(t_{n-2}) + i \cdot u_1 x(t_{n-2})} \right] \\ &= e^{A(u_1, 0, \Delta t_n)} \cdot e^{A(u_1, u_2(t_{n-1}), \Delta t_{n-1})} \cdot \mathbb{E} \left[e^{i \cdot (-i \cdot B(u_1, u_2(t_{n-1}), \Delta t_{n-1}) \cdot v(t_{n-2}) + u_1 x(t_{n-2}))} \right] \\ &= \dots \end{aligned}$$

$$= e^{\sum_{i=1}^N A(u_1, u_2(t_{n-i}), \Delta t_{N-1}) + B(u_1, u_2(t_1), \Delta t_1) \cdot v(t_0) + i \cdot u_1 x(t_0)}, \quad (61)$$

where $u_2(t_{j-1}) = -i \cdot B(u_1, u(t_j), \Delta t_j)$ with $u(t_n) = 0$. Therefore one can compute the characteristic function by bootstrapping over the time intervals where the parameters are constant. As for the continuity of the characteristic function (61), Afshani [AS09] verified that for $\alpha \in [-1, 0]$ one can safely use the formulation in equation (45). If one wishes to use more advanced interpolation methods for the time dependent model parameters one can no longer apply the tower law to derive an analytical solution for the characteristic function, but has to solve the system of Ricatti equations (29) and (30) numerically. A standard method to solve this problem is to use a Runge-Kutta solver [PTVF92] for the ordinary differential equation.

Forward starting options

The payoff of a forward starting option fixing at t and maturing at T is given by

$$\left(\frac{S(T)}{S(t)} - \omega \right)^+, \quad (62)$$

with strike ω . In order to value this deal it suffices to know the characteristic function the forward characteristic function, as pointed out by [Hon04]. In fact, the characteristic function of the ratio of the underlying is all it takes. As an example, consider the setting where we have both the asset and a stochastic volatility v as the two state variables. Denoting $\log(S(T)/S(t)) = x(T) - x(t)$, we obtain

$$\begin{aligned} \mathbb{E} \left[e^{i \cdot u_1 (x(T) - x(t))} \right] &= \mathbb{E} \left[e^{-i \cdot u_1 x(t)} \mathbb{E} \left[e^{i \cdot u_1 x(T)} \mid \mathcal{F}_t \right] \right] \\ &= \mathbb{E} \left[e^{-i \cdot u_1 x(t)} \cdot \varphi(u_1, 0, \tau, x(t), v(t)) \right] \\ &= \mathbb{E} \left[e^{-i \cdot u_1 x(t)} \cdot e^{A(u_1, 0, \tau) + B(u_1, 0, \tau) \cdot v(t) + i \cdot u_1 x(t)} \right] \\ &= e^{A(u_1, 0, \tau)} \cdot \mathbb{E} \left[e^{B(u_1, 0, \tau) \cdot v(t)} \right] \\ &= e^{A(u_1, 0, \tau)} \cdot \mathbb{E} \left[e^{i \cdot (-i \cdot B(u_1, 0, \tau) \cdot v(t))} \right] \\ &= e^{A(u_1, 0, \tau)} \cdot \varphi(0, -i \cdot B(u_1, 0, \tau), t, v(0), x(0)) \\ &= e^{A(u_1, 0, \tau)} \cdot e^{A(0, -i \cdot B(u_1, 0, \tau), t) + B(0, -i \cdot B(u_1, 0, \tau), t) \cdot v(0)}. \end{aligned} \quad (63)$$

Thus pricing forward starting options with a known (multivariate-) characteristic function, is numerically not more expensive than pricing standard European options.

3 Numerical Fourier inversion

In this section we discuss how to implement the semi-finite integral in equations (9) and (11) numerically. Several different methods have been suggested in the literature in recent years and we are going to compare the Fast Fourier Transformation (FFT) approach of Carr and Madan [CM99], saddlepoint approximations as well as optimal contour integration methods [LK07].

3.1 Fast Fourier transformation

The fast Fourier transformation, as first introduced by Gauss and reinvented by Cooley and Tukey [CT65], allows to compute efficiently sums of the form

$$\omega_m = \sum_{j=1}^N e^{2\pi i \cdot (j-1)(m-1)/N} x_k, \quad m = 1, \dots, N, \quad (64)$$

with a complexity of $\mathcal{O}(N \log N)$. Discretising the semi-finite integral (11) with equidistant quadrature roots $\Delta x = x_{i+1} - x_i$,

$$\int_0^\infty e^{-ivk} \psi_\alpha(v) dv \approx \sum_{j=1}^N e^{-i \cdot (j-1) \cdot \Delta x \cdot k} \psi_\alpha(x_j) \quad (65)$$

thus allows to compute simultaneously option prices for a range of N strikes. Hence we choose

$$k = -\frac{N \cdot \Delta k}{2} + (m-1) \cdot \Delta k, \quad m = 1, \dots, N. \quad (66)$$

Here the discretisation in (log)-strike domain and the quadrature discretisation have to obey the Nyquist relation

$$\Delta k \cdot \Delta x = 2\pi/N, \quad (67)$$

which effectively restricts the choice of the integration domain and the range of strikes to be computed. Therefore we have to balance the interpolation error in strike domain against the quadrature error. For a more detailed analysis we refer the interested reader to [Lee05].

3.2 Saddle-point approximation

Daniels [Dan54] first used the concept of saddlepoint approximations in the context of calculating probability densities via Fourier inversion

$$p(x) = \frac{1}{2\pi} \int_{-\infty-i\alpha}^{\infty-i\alpha} e^{-izx} \varphi_p(z) dz, \quad (68)$$

where φ_p is the characteristic function associated with the density p . The standard inversion formula would set α equal to zero and from an analytical point of view the choice of α is irrelevant anyway. From a numerical perspective the quadrature problem (68) we however get a distinctively different behaviour of the integrand depending on the choice of α . Saddlepoint approximations typically continue along the following lines. Let $M_{\text{saddle}}(z) = \log \varphi_p(z)$, the characteristic exponent, then Daniels advocated to choose α as the minimum of the damped characteristic exponent (80):

$$\alpha_s^* = \underset{\alpha \in \{\alpha_{\text{Min}}, \alpha_{\text{Max}}\}}{\text{argmin}} \quad -\alpha x + M_{\text{saddle}}(-i\alpha). \quad (69)$$

Thus we obtain $M'_{\text{saddle}}(-i\alpha_s^*) = ix$. Applying a Taylor expansion around its minimum leads to

$$M_{\text{saddle}}(z) - izx = M_{\text{saddle}}(-i\alpha_s^*) - \alpha_s^* x + \frac{1}{2} M''_{\text{saddle}}(-i\alpha_s^*) (z + i\alpha_s^*)^2 + \mathcal{O}(z^3), \quad (70)$$

so that the density can be approximated via

$$\begin{aligned} p(x) &= \frac{1}{2\pi} \int_{-\infty-i\alpha_s^*}^{\infty-i\alpha_s^*} e^{-izx} \varphi_p(z) dz \approx \frac{\varphi_p(-i\alpha_s^*) e^{-\alpha_s^* x}}{2\pi} \int_{-\infty-i\alpha_s^*}^{\infty-i\alpha_s^*} e^{\frac{1}{2} M''_{\text{saddle}}(-i\alpha_s^*) z^2} dz \\ &= \frac{\varphi_p(-i\alpha_s^*) e^{-\alpha_s^* x}}{\sqrt{-2\pi M''_{\text{saddle}}(-i\alpha_s^*)}} =: p_{\text{simple}}(x). \end{aligned} \quad (71)$$

This ‘‘simple’’ saddlepoint approximation reduces the quadrature problem (68) to a few function evaluations. As pointed out by Ait-Sahalia and Yu [ASY06], the name ‘‘saddlepoint’’ stems from the shape of the right-hand side of (70) in a neighbourhood of its minimum, which can be seen as a saddle (see figure 1). Further one can use this saddlepoint approach to derive an approximation for the cumulative density, a possible approach being the Lugannani-Rice formula [LR80].

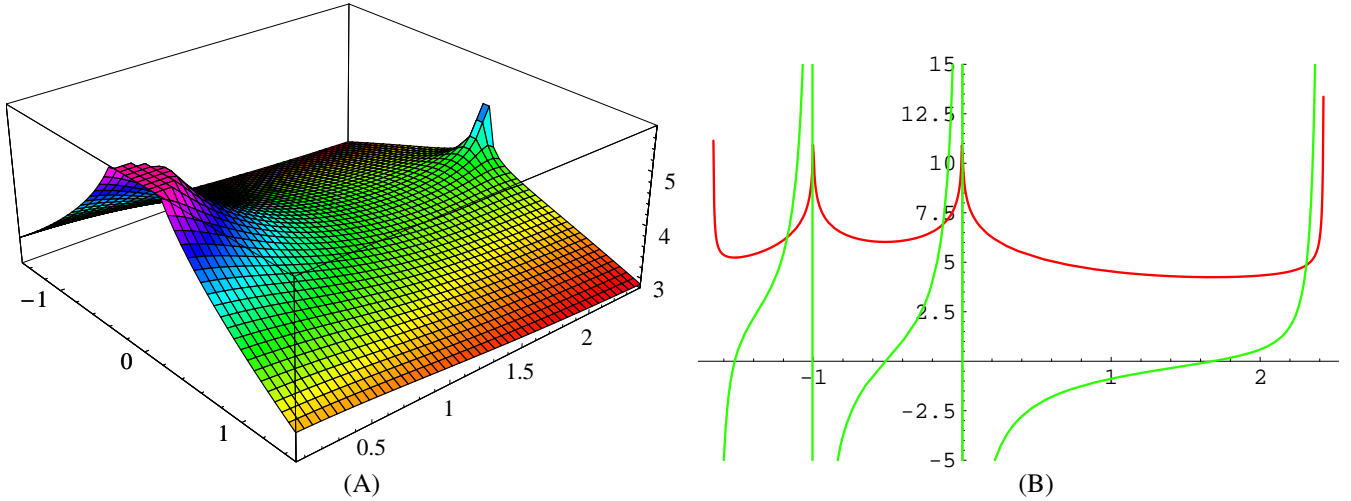


Figure 1: Simple saddlepoint of the option price in the Heston model. (A) plot of the characteristic function times the Fourier transform of the payoff in the complex plane $z = x + iy$. (B) $x = 0$. The parameters in the Heston model are $\kappa = 0.2$, $\omega = 0.228$, $\rho = -0.511$, $\tau = 20$, $f = 1$, $k = 1.1$, $v_0 = 0.028$ and $\theta = 0.028$.

Whilst saddlepoint approximation work remarkably well in the tails of the distribution they typically fail to produce adequate results for pricing options around the money. Rogers and Zane [RZ98] applied the Lugannani-Rice formula to compute the probabilities Π_1 (3) and Π_2 (4). As expected, they obtained accurate results for options close to maturity and away from the at-the-money level, though the accuracy was lower around the at-the-money level. For small option prices this approach will lead to cancellation errors, so that we suggest to apply the saddlepoint approximation directly to the Carr-Madan representation (11). Instead of expanding the characteristic exponent around its minimum, $M_{\text{call-price}}$ here has to be taken equal to:

$$M_{\text{call-price}}(z) = \frac{1}{2} \log \psi_{\alpha}(z)^2. \quad (72)$$

Let us denote the corresponding saddle with α_c^* , leading to a “simple” saddlepoint approximation of the option price

$$\begin{aligned} C(S, K, \tau, \alpha_c^*) &= R(S, K, \alpha_c^*) + \frac{1}{2\pi} \int_{-\infty - i\alpha_c^*}^{\infty - i\alpha_c^*} e^{-izx} \psi_{\alpha_c^*}(z) dz \\ &\approx R(S, K, \alpha_c^*) + \frac{\psi(-i\alpha_c^*) e^{-\alpha_c^* k}}{\sqrt{-2\pi M_{\text{call-price}}''(-i\alpha_c^*)}} =: C_{\text{simple}}(x). \end{aligned} \quad (73)$$

In figure 1, we plot the characteristic function to show that it indeed resembles the shape of a saddle. Naturally, higher-order expansions of the density can also be derived. Ait-Sahalia and Yu give a higher-order expansion derived by expanding the left-hand side of (70) to fourth order in z :

$$p_{\text{higher-order}}(x) = p_{\text{simple}}(x) \left(1 + \frac{1}{8} \frac{M_{\text{saddle}}^{(4)}(-i\alpha_c^*)}{\left(M_{\text{saddle}}^{(2)}(-i\alpha_c^*)\right)^2} - \frac{5}{24} \frac{\left(M_{\text{saddle}}^{(3)}(-i\alpha_c^*)\right)^2}{\left(M_{\text{saddle}}^{(2)}(-i\alpha_c^*)\right)^3} \right), \quad (74)$$

where $M_{\text{call-price}}^{(i)}$ denotes the i -th derivative. We will refer to the analogue of (74) in the context of option pricing as the “higher-order” saddlepoint approximation of the option price.

As a final note, we mention that the saddlepoint approximations considered here can be seen as an expansion around a Gaussian base (the term $\frac{1}{2}z^2$ in (70)). Ait-Sahalia and Yu show how to derive saddlepoint approximations which expand the characteristic exponent around a non-Gaussian base, and demonstrate that this improves the accuracy of the approximations considerably for jump-diffusion and Lévy

models. Though this route is viable, the problem of determining which base is most suitable is obviously a very model-dependent problem. As we would like to keep our pricing methods as model-independent as possible, the only model-dependent part being of course the specification of the characteristic function and a transformation function, we only discuss the saddlepoint approximations around a Gaussian base. In case the infinite domain of integration cannot be transformed to a finite one, we choose an appropriate upper limit of integration.

3.3 Optimal contour integration

In this section we show that combining the ideas of saddlepoint approximation with numerical quadrature schemes provides a very efficient and in particular highly accurate and reliable method for Fourier inversion problems [LK07, Sta09]. This is of great importance since the pricing of European options via Fourier inversion is typically used in calibrating the model parameters via non-linear optimisation routines to match market prices, necessitating a robust approach that works for a wide range of possible parameter values.

Analysing the semi-finite integral in equation (9) and (11) respectively one has to consider the quadrature error, the truncation error and the choice of the integration contour α . Lee [Lee05] discusses these choices in the context of a Discrete Fourier Transform (DFT), and suggests a minimisation algorithm to determine the parameters of the discretisation. Transforming the semi-finite integral to a finite domain does entirely remove the truncation error and adaptive quadrature schemes, such as an adaptive Gauss-Lobatto, offer an efficient and reliable approach to evaluate the integral numerically. Thus the only remaining choice is the dampening factor α .

Choosing α too small or too big leads to problems with either cancellation errors or highly oscillating integrands. Ideally one wishes to choose α such that the integrand is as constant as possible over the whole integration area. Förster and Petras [FP91] have shown that the oscillation of an integrand, as measured by total variation (here on the finite interval $[-1, 1]$)

$$\text{TV}(f) = \int_{-1}^1 \left| \frac{\partial f}{\partial x}(x) \right| dx . \quad (75)$$

can be directly linked to estimate the approximation error of a numerical integration scheme. Here we can use α to minimize the total variation of the (rescaled-)integrand¹

$$\alpha^* = \underset{\alpha \in \{\alpha_{\min}, \alpha_{\max}\}}{\text{argmin}} e^{-\alpha k} \int_0^\infty \left| \frac{\partial \psi(v, \alpha)}{\partial v} \right| dv . \quad (76)$$

Clearly this optimization problem is of a rather theoretical nature as it has to be solved again for each option price and would therefore require many more function evaluations than the original problem.

Assuming that the function $\psi(v, \alpha)$ is monotone in v on $[0, \infty)$ when choosing an optimal α , equation (76) simplifies to

$$e^{-\alpha k} \int_0^\infty \left| \frac{\partial \psi(v, \alpha)}{\partial v} \right| dv = e^{-\alpha k} |\psi(0, \alpha) - \psi(\infty, \alpha)| = |e^{-\alpha k} \psi(0, \alpha)| , \quad (77)$$

since the characteristic function of an affine jump-diffusion vanishes in infinity. Thus the optimal α is given by

$$\alpha_{\text{payoff-dependent}}^* = \underset{\alpha \in \{\alpha_{\text{Min}}, \alpha_{\text{Max}}\}}{\text{argmin}} |e^{-\alpha k} \psi(0, \alpha)| , \quad (78)$$

¹Rescaling is required to facilitate comparison of total variation for different values of α .

or equivalently

$$\alpha_{\text{payoff-dependent}}^* = \underset{\alpha \in \{\alpha_{\text{Min}}, \alpha_{\text{Max}}\}}{\operatorname{argmin}} \left[-\alpha k + \frac{1}{2} \log(\psi(0, \alpha)^2) \right] =: \Psi(\alpha, k), \quad (79)$$

rendering the calculation more stable. Finding the optimal α thus requires finding the minimum of Ψ in equation (79). The optimal α resulting from this optimisation problem will be referred to as the payoff-dependent α , as ψ depends on the particular payoff function we are considering. In the following we will also consider a payoff-independent alternative:

$$\alpha^* = \underset{\alpha \in \{\alpha_{\text{Min}}, \alpha_{\text{Max}}\}}{\operatorname{argmin}} [-\alpha k + \log(\varphi(-(\alpha + 1)i))] =: \Phi(\alpha, k). \quad (80)$$

Coincidentally, this payoff-independent way of choosing α has a close link to saddlepoint approximation since solving equation (80) is equal to (69). Choudhury and Whitt [CW97] consider a similar contour shift to avoid numerical problems in the numerical inversion of non-probability transforms. Solving equation (79) is equivalent to

$$\frac{\partial \Psi(\alpha, k)}{\partial \alpha} = 0 \quad (81)$$

and we illustrate its typical shape in figure 2. A good rule of thumb to find the optimal value of α , is to assume that $\alpha^* \in (0, \alpha_{\text{Max}})$ for $F < K$ and $\alpha^* \in (\alpha_{\text{Min}}, -1)$ for $F > K$, i.e. to price out-of-the money options.

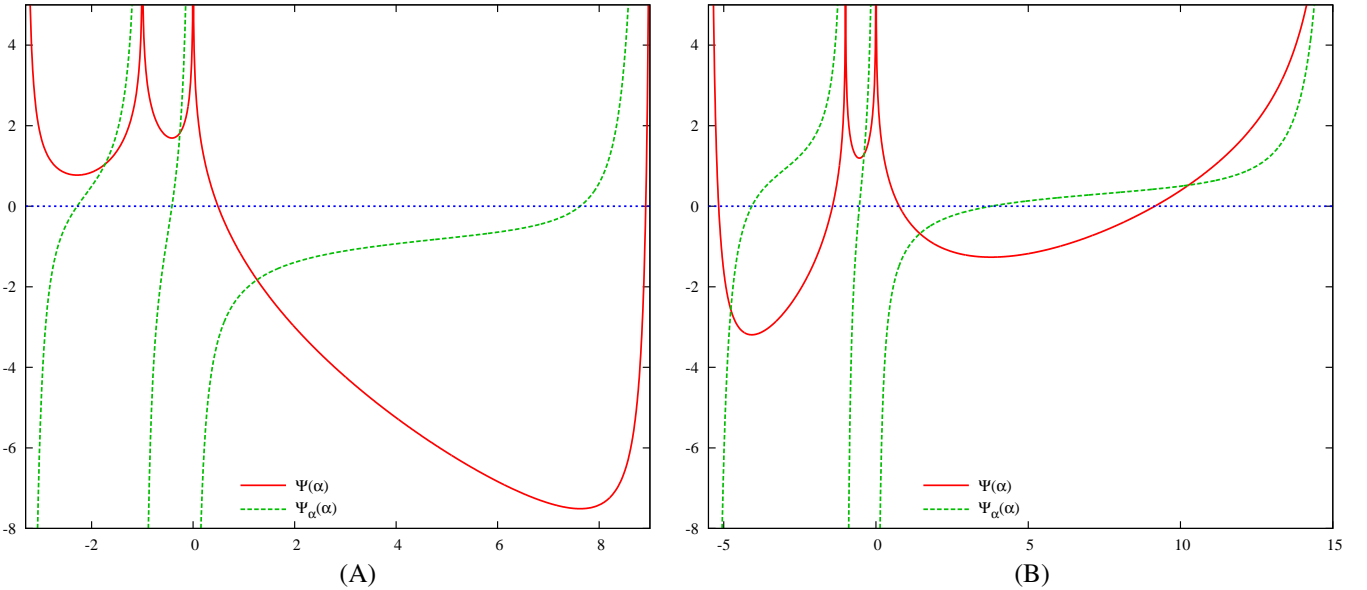


Figure 2: Function Ψ given by equation (79) and its first derivative for the Heston model w.r.t. α . Underlying: $dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_S(t)$ with $S = F = 1$ and $\mu = 0$. Variance: $dV_t = \kappa(\theta - V_t)dt + \omega\sqrt{V_t}dW_V(t)$ with $V_0 = \theta = 0.1$, $\kappa = 1$, $\omega = 1$ and $\rho = -0.7$. (A): $\tau = 1$ and $K = 1.2$. (B): $\tau = 1/2$ and $K = 0.7$.

Lord and Kahl [LK07] provide further analysis and a detailed comparison on the different Fourier inversion methods. Numerical tests in particular show the superiority of the optimal contour integration method in terms of stability and robustness, as well as reducing the number of quadrature points required to approximate the semi-finite integral (9) to a certain accuracy and thus improving the computational speed.

We conclude this section with presenting two models which allow for a closed-form expression of the payoff-independent α (80).

3.3.1 Black-Scholes model

The simplicity of the characteristic function of the Black-Scholes model allows to compute the optimal payoff-independent α analytically. We find:

$$\alpha^* = \operatorname{argmin}_{\alpha \in \mathbb{R}} \Phi(\alpha, k) = -\frac{f - k + \frac{1}{2}\widehat{\eta}^2\tau}{\widehat{\eta}^2\tau} = -\frac{d_1}{\widehat{\eta}\sqrt{\tau}}, \quad (82)$$

where d_1 is a well-known part of the Black-Scholes option pricing formula. Here it is worth noting that the payoff-dependent α^* can also be solved in (semi)-closed-form, as the solution of a fourth-order polynomial in α .

3.3.2 Variance Gamma model

The Variance Gamma model, as introduced in section 2.1.1, is one other model which allows for an analytic expression of the payoff-independent α . Standard yet tedious calculations yield:

$$\alpha^* = \operatorname{argmin}_{\alpha \in \mathbb{R}} \Phi(\alpha, k) = -\frac{\theta}{\sigma^2} - 1 + \frac{\tau}{\nu\widetilde{m}} - \operatorname{sgn}(\widetilde{m})\sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\nu\sigma^2} + \frac{\tau^2}{\nu^2\widetilde{m}^2}} \quad (83)$$

where we introduced $\widetilde{m} = \widetilde{f} - k = f - k + \omega\tau$, a quantity related to the log-moneyness of the option. It is worth mentioning that Ait-Sahalia and Yu [ASY06] provide the saddlepoint of the Variance Gamma model and several other models where it can be calculated analytically.

3.4 Pricing early exercise options

In this section we will expand on the brief overview in section 1.2 by detailing the calculations required for the CONV method by [LFB08]. Taking the continuation value C from (12), and assuming deterministic interest rates for convenience, we can write

$$C(t_m, x) = P(t_m, t_{m+1}) \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|x) dy. \quad (84)$$

Here $f(y|x)$ denotes the conditional probability density for the asset. The coordinates do not have to represent the asset price itself, but could be monotone functions thereof. In order to be able to speed up the calculations by using the FFT to approximate such convolutions, the main premise of the CONV method is that the conditional probability density only depends on x and y via their difference, i.e. $f(y|x) = f(y - x)$. As one of the defining property of Lévy processes is that its increments are independent of each other, this naturally holds for this class of processes.

Using similar arguments to section 1.1 we rewrite the previous equation as follows

$$C(t_m, x) = \frac{e^{-\alpha x} P(t_m, t_{m+1})}{\pi} \int_0^{\infty} \operatorname{Re} \left[e^{-ivx} \varphi(-(v - i\alpha)) \cdot \widetilde{V}_\alpha(t_{m+1}, v) \right] dv. \quad (85)$$

where

$$\widetilde{V}_\alpha(t, v) = \mathcal{F}\{e^{\alpha y} \cdot V(t, y)\}(v) = \int_{-\infty}^{\infty} e^{iuy} e^{\alpha y} V(t, y) dy \quad (86)$$

From here on we will denote dampened quantities by a subscript α , and Fourier transforms by using a superscript \sim . We have assumed that the characteristic function of the conditional probability density is known in closed-form. The two main differences with the Carr-Madan approach in [CM99] is firstly

that the transform of the payoff function is not known in closed-form. Secondly, we take a transform with respect to the log-spot price instead of the log-strike price, something which [Lew01] and [Rai00] also consider for European option prices. The damping factor is again necessary when considering e.g. a Bermudan put, as then $V(t_{m+1}, x)$ tends to a constant when $x \rightarrow -\infty$, and as such is not L^1 -integrable. However, as we will see later on we do not strictly need a damping factor when we discretise this equation, which is beneficial as we then do not have to analyse the poles of all payoff functions. For this reason we have omitted the residue term in (85).

The algorithm from equation (12) in pseudo code now becomes:

```

 $V(t_M, x) = E(t_M, x)$  for all  $x$ 
 $E(t_0, x) = 0$  for all  $x$ 
for  $m = M - 1$  to  $0$  do
  Dampen  $V(t_{m+1}, x)$  with  $\exp(\alpha x)$  and take its Fourier transform
  Calculate  $C(t_m, x)$  by means of equation (85)
  Set  $V(t_m, x) = \max \{ (E(t_m, x), C(t_m, x)) \}$ 
end for

```

Algorithm 1: The CONV algorithm for Bermudan options

For American options one can use Richardson extrapolation, for which we refer the reader to [CCS02]. We will detail how the above algorithm is implemented in [LFBO08]. In order to be able to employ the use of the FFT, we need to introduce uniform grids for v , x and y

$$u_j = u_0 + j\Delta u, \quad x_j = x_0 + j\Delta x, \quad y_j = y_0 + j\Delta y, \quad (87)$$

where $j = 0, \dots, N - 1$. Though they may be centered around a different point, the x - and y -grids have the same mesh size: $\Delta x = \Delta y$. Further, the Nyquist relation (67) must be satisfied, i.e.,

$$\Delta u \cdot \Delta y = \frac{2\pi}{N}. \quad (88)$$

Let us now approximate (86) with a general Newton-Cotes rule, and (85) with a left-rectangle rule². Rewriting the equation slightly yields

$$\frac{C_\alpha(x_p, t_m)}{P(t_m, t_{m+1})} \approx \frac{\Delta u \Delta y}{2\pi} \sum_{j=0}^{N-1} e^{-iu_j x_p} \phi(-(u_j - i\alpha)) \sum_{n=0}^{N-1} w_n e^{iu_j y_n} V_\alpha(y_n, t_{m+1}). \quad (89)$$

To be able to directly implement this with the Discrete Fourier Transform (DFT), let us denote the DFT and its inverse of a sequence x as

$$\mathcal{D}_j\{x_n\} := \sum_{n=0}^{N-1} e^{ijn2\pi/N} x_n, \quad \mathcal{D}_n^{-1}\{x_j\} = \frac{1}{N} \sum_{j=0}^{N-1} e^{-ijn2\pi/N} x_j. \quad (90)$$

Setting $u_0 = -N/2\Delta u$, which we clarify later, and noticing that $e^{inu_0\Delta y} = (-1)^n$ allows us to rewrite equation (89) as

$$\frac{C_\alpha(x_p, t_m)}{P(t_m, t_{m+1})} \approx e^{iu_0(y_0 - x_0)} (-1)^p \mathcal{D}_p^{-1} \{ e^{ij(y_0 - x_0)\Delta u} \phi(-(u_j - i\alpha)) \mathcal{D}_j \{ (-1)^n w_n V_\alpha(y_n, t_{m+1}) \} \}. \quad (91)$$

We will now detail the grid choice used in [LFBO08]. There, the following grids were chosen

$$u_j = (j - \frac{n}{2})\Delta u, \quad x_j = \epsilon_x + (j - \frac{L}{2})\Delta y, \quad y_j = \epsilon_y + (j - \frac{L}{2})\Delta y, \quad (92)$$

²We will clarify the discrepancy in integration methods later.

where $\epsilon_x = d_m - \lceil d_m/\Delta x \rceil \cdot \Delta x$ and ϵ_y is chosen in a similar fashion. By using a different grid for the payoff transform (y), and the inverse Fourier transform of the continuation value (x), the user is free to place one discontinuity on the grid at each time slice. In the x -grid, d_m is the discontinuity which will be placed on the grid. This is important to ensure smooth convergence. For a simple European call for example, one would want to ensure that at the final time slice the strike price lies on the grid, whereas at the initial time slice one would want the current spot price to lie on the grid. To enhance the convergence of Bermudan options in the CONV algorithm, the CONV algorithm estimates the location of the exercise boundary and places this on the grid in a second iteration, see [LFBO08, Section 4.4]. Unfortunately, due to the restrictions of a uniform grid, it is impossible to place more discontinuities on the grid.

With regards to step sizes, [LFBO08] have chosen $\Delta y = L/N$. The Nyquist relation in (88) implies $\Delta u = 2\pi/L$. This brings us to the final parameter, L . As this determines the span of the discretisation in the asset direction, it is typically chosen as a multiple of the standard deviation of the underlying at that timeslice. Due to the inverse relation between the grid in the asset domain and the Fourier domain, there is a trade-off in the choice of L . While a larger value of L implies smaller truncation errors, it also causes the range of the grid in the Fourier domain to be smaller.

It can be shown that the left-rectangle rule does not affect the order of convergence in (89). The convergence depends on some interaction between the chosen Newton-Cotes rule (for the payoff transform), and the speed of decay of the characteristic function, see [LFBO08] for more details. It suffices to say that for the Black-Scholes model, the order of convergence is determined by the Newton-Cotes rule, whereas for Lévy models where the characteristic function decays as a power, as is the case in the VG model, the order can be smaller than this for short maturities.

We should mention the damping coefficient α . The damping coefficient is not strictly necessary in the discrete setup, as we have essentially replaced L^1 -integrability over an infinite interval with L^1 -summability on a finite one. Nevertheless, it makes sense to adhere to the guidelines from the continuous time setup, as the functions will resemble its continuous counterparts more and more as the accuracy increases. The CONV method shows the same dependence on the damping coefficient as we have found in section 3.3.

The difference between the COS method and the CONV method lies in the computation of the continuation value. In the COS method of [FO08a, FO08b], both the density and the value function in (84) are replaced by their respective Fourier-cosine expansions. We remind the reader that the Fourier-cosine expansion of a function f on $[a, b] \in \mathbb{R}$ is given by

$$f(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cdot \cos\left(k\pi \frac{x-a}{b-a}\right) \quad (93)$$

where

$$A_k = \frac{2}{b-a} \int_a^b f(x) \cos\left(k\pi \frac{x-a}{b-a}\right) dx \quad (94)$$

For plain vanilla European options the strategy is to explicitly calculate the Fourier-cosine coefficients in (94) for both the payoff function and the density function. This case is dealt with in [FO08a]. The coefficients can be arranged conveniently such that once again the FFT can be employed to calculate the prices for a uniform grid of log-asset prices³. The convergence depends on the decay rate of the cosine series coefficients. For plain vanilla calls and puts this is dictated by the rate of decay of the characteristic function. Practically, this means that for models such as the Black-Scholes model, the convergence is exponential, whereas for the VG model, the convergence is geometric, and actually a function of the time to maturity (once again leading to very slow convergence for very short maturities).

³As in the CONV method, this only holds for exponential Lévy models.

For Bermudan options, additional work needs to be done as the value function V is no longer known analytically. The paper [FO08b] deals with standard Bermudan puts or calls in one-dimensional models, where there is a unique exercise boundary. Above or below this asset level, it will be optimal to exercise. Once we know this exercise boundary, the payoff can be split into two parts - for one we can calculate the Fourier-cosine series analytically (as in the European case). For the remainder, [FO08b] shows how to calculate them numerically. The only problem is the exact location of the exercise boundary. As the continuation value in (84) can be calculated efficiently in one point in the COS method, i.e. without having to evaluate it in all gridpoints, the authors of [FO08b] have chosen to first solve the exercise boundary using Newton's method. The convergence of the COS method for Bermudans is, as in the European case, dominated by the decay rate of the cosine series coefficients. We refer the interested reader to both aforementioned papers for more details.

One final detail worth mentioning of Fourier-based calculations methods is the calculation of Greeks. As differentiation is exact in Fourier space, Greeks can be obtained in both the CONV and COS methods at virtually no extra cost. To give the reader an idea of how this would work, we sketch how one would calculate the Δ of an option, i.e. $\Delta = \frac{\partial V}{\partial S}$. We have:

$$\mathcal{F}\{e^{\alpha x}V(t_0, x)\} = P(t_0, t_1)A(u) \quad (95)$$

where we defined $A(u) := \mathcal{F}\{e^{\alpha y}V(t_1, y)\} \cdot \phi(-u + i\alpha)$. If the second argument of V is the logarithm of the asset, then simple differentiation shows that the Δ can be obtained as

$$\Delta = \frac{e^{-\alpha x} \cdot P(t_0, t_1)}{S} \left[\mathcal{F}^{-1}\{-iuA(u)\} - \alpha \mathcal{F}^{-1}\{A(u)\} \right], \quad (96)$$

The additional calculations only have to be performed at the final timestep; we see that in order to calculate the Δ of an option, we only require one additional Fourier transform.

$(N = 2^n)$	Black-Scholes			Variance Gamma		
	time(msec)	error	conv.	time(msec)	error	conv.
7	0.23	-2.72e-2	-	0.28	-9.63e-2	-
8	0.46	-7.36e-3	3.7	0.55	-1.07e-2	9.0
9	0.90	-2.00e-3	3.7	1.09	-2.27e-3	4.7
10	2.00	-5.22e-4	3.8	2.15	-6.06e-4	3.8
11	3.85	-1.32e-4	4.0	4.38	-1.59e-4	3.8
12	7.84	-3.31e-5	4.0	9.29	-4.08e-5	3.9

Table 1: CPU time, error and convergence rate pricing a 10-times exercisable Bermudan put under the Black-Scholes and VG models; $K = 110$, $T = 1$ and reference values of 11.98745352 and 9.040646119 respectively;

We conclude with a numerical example of the CONV method from [LFBO08, Section 5.3]. Here we have priced a 10-times exercisable Bermudan put under both the Black-Scholes and VG models. For the Black-Scholes model, we used a spot price of 100, a risk-free rate of 10%, and a volatility of 25%. For the VG model, the spot price and risk-free rate are the same; the parameters $\sigma = 0.12$, $\theta = -0.14$, $\nu = 0.2$. In Table 1 we used Discretisation II, which corresponds to the grid advocated in (92).

As one can see, the convergence is approximately second order, as predicted by the theory. The convergence is fairly regular, which greatly facilitates the use of extrapolation to arrive at e.g. American option prices. The computational speed of the method is highly satisfactory; the difference in computational speed between both models arises from the fact that the VG characteristic function is slightly more computationally intensive than the Black-Scholes one.

4 Summary

This article provides an overview on Fourier option pricing in computational Finance and shows how to derive the characteristic function for affine (jump) diffusion and Lévy models. Special emphasis is the numerical implementation of the semi-finite integral when pricing European, forward starting and early exercise options. In particular we compare the fast Fourier transformation, saddlepoint approximations and optimal contour integration methods. For European (and forward starting) option pricing the optimal contour integration method is highly efficient and robust and thus the ideal choice for a Black-Box algorithm. We further discuss the application of Fourier inversion methods in the context of American and Bermudan option pricing and present the CONV method in greater detail.

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