



# *Fifty shades of SABR simulation*

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The material presented here is joint work with Adam Farebrother.  
The forthcoming paper will be posted on  
<http://www.rogerlord.com> in the near future:

LORD, R. AND A. FAREBROTHER (2015). “Fifty shades of SABR simulation”.

# *Part I – SABR model and schemes*

# SABR model

In the SABR (stochastic  $\alpha\beta\rho$ ) model, due to Hagan, Kumar, Lesniewski and Woodward [2002] the (forward) asset follows a CEV process with a lognormal stochastic volatility:

$$dS(t) = \sigma(t) S(t)^\beta dW_S(t)$$

$$d\sigma(t) = \alpha \sigma(t) dW_\sigma(t)$$

where  $dW_S(t) \cdot dW_\sigma(t) = \rho dt$ , and:

- ▶  $\alpha$  is the volatility of volatility;
- ▶  $\beta$  is the elasticity of variance or *backbone*.

# SABR model (2)

Some probabilistic results:

- ▶ For  $\frac{1}{2} \leq \beta < 1$ , a unique solution exists and the zero boundary is absorbing;
- ▶ For  $\beta < \frac{1}{2}$  the SDE does not have a unique solution, meaning that a boundary condition has to be imposed;
- ▶ The SABR expansion formula is consistent with an absorbing boundary at zero;
- ▶ All positive moments are finite, following Andersen and Piterbarg [2006].

# *SABR schemes*

Like in the Schöbel-Zhu model, simulating the volatility is not an issue, as it follows a GBM. The challenge of any scheme is how to combine this with a simulation of the asset. We consider:

- ▶ Schemes based on relation of SABR to a squared Bessel process (Islah [2009]);
- ▶ Log-Euler schemes, a la Andersen and Brotherton-Ratcliffe [2005] or Lord et al. [2010];
- ▶ Ninomiya-Victoir scheme with drift, see Bayer, Friz and Loeffen [2013].

# SABR schemes (2)

Islah [2009] has shown that SABR is closely related to a squared Bessel process. Consider  $X(t) = \frac{1}{1-\beta} S(t)^{1-\beta}$ :

$$dX(t) = -\frac{\beta\sigma(t)^2}{2(1-\beta)X(t)} dt + \sigma(t) \left( \rho dW_\sigma(t) + \sqrt{1-\rho^2} dZ(t) \right)$$

From the variance equation we obtain:

$$\sigma(t) = \sigma(0) + \alpha \int_0^t \sigma(u) dW_\sigma(u)$$

leading to:

$$\begin{aligned} X(t) &= X(0) + \frac{\rho}{\alpha} (\sigma(t) - \sigma(0)) - \int_0^t \frac{\beta\sigma(u)^2}{2(1-\beta)X(u)} du \\ &\quad + \sqrt{1-\rho^2} \int_0^t \sigma(u) dZ(u) \end{aligned}$$

# SABR schemes (3)

If we condition on  $\sigma(t)$ , we could introduce a shifted process  $\tilde{X}(t)$  with initial condition:

$$\tilde{X}(0) = X(0) + \frac{\rho}{\alpha} (\sigma(t) - \sigma(0))$$

and approximate  $\tilde{X}(t) \approx X(t)$ , which is exact for  $\rho = 0$ .

Now, if we move to  $Y(t) = \tilde{X}(t)^2$ , we obtain:

$$dY(t) = 2\sqrt{Y(t)} \sqrt{1-\rho^2} \sigma(t) dZ(t) + \delta(1-\rho^2) \sigma(t)^2 dt$$

where  $\delta = \frac{1-2\beta-\rho^2(1-\beta)}{(1-\beta)(1-\rho^2)}$ .

# SABR schemes (4)

Finally, we apply a time change:

$$v(t) = \sqrt{1 - \rho^2} \int_0^t \sigma(u)^2 du$$

(and thus condition on the integrated variance):

$$dY(v(t)) = 2\sqrt{Y(v(t))} dZ(v(t)) + \delta dv(t)$$

The derivation shows that, conditional on  $\sigma(t)$  and the integrated variance, the transformed asset follows a time-changed squared Bessel process, with an adjusted initial condition. For  $\rho = 0$  this is exact, and otherwise an approximation.

# SABR schemes (5)

The suggested algorithm therefore is:

1. Simulate  $\sigma(t) | \sigma(0)$
2. Simulate  $\int_0^t \sigma(u)^2 du | \sigma(t), \sigma(0)$
3. Simulate  $Y(v(t)) | Y(0), \sigma(t), \sigma(0), \int_0^t \sigma(u)^2 du$ , in:

$$dY(t) = 2\sqrt{Y(t)} dZ(t) + \frac{1 - 2\beta - \rho^2(1 - \beta)}{(1 - \beta)(1 - \rho^2)} dt$$

and:

$$Y(0) = \left( \frac{1}{1-\beta} S(0)^{1-\beta} + \frac{\rho}{\alpha} (\sigma(t) - \sigma(0)) \right)^2$$

$$v(t) = \sqrt{1 - \rho^2} \int_0^t \sigma(u)^2 du$$

## SABR schemes (6)

An alternative for moment-matching schemes is to immediately consider  $Y(t) = \frac{1}{(1-\beta)^2} S(t)^{2(1-\beta)}$ :

$$dY(t) = 2\sigma(t)\sqrt{Y(t)} dW_S(t) + \delta\sigma(t)^2 dt$$

and use the time-change  $v(t) = \int_0^t \sigma(u)^2 du$ . Since:

$$W_\sigma(v(t)) - W_\sigma(v(t)) \stackrel{d}{=} \alpha(\sigma(t) - \sigma(0))$$

this opens up ways to embed the dependence with the variance process. We leave this for further research.

# *SABR schemes (7)*

In the remainder we focus on the ingredients involved:

- ▶ Approximating the integrated variance process
- ▶ Moment-matching schemes for the asset
- ▶ Exact simulation of the asset

We also consider other schemes. Like Andersen and Brotherton-Ratcliffe [2005] or Lord [2010] we could switch to log-coordinates:

$$d \ln S(t) = -\frac{1}{2} \sigma(t)^2 S(t)^{2(\beta-1)} dt + \sigma(t) S(t)^{\beta-1} dW_S(t)$$

and use this transformation within an Euler scheme.

# SABR schemes (8)

We could also choose to use the information of the volatility process by only freezing the asset:

$$\begin{aligned}\ln S(t) = & \ln S(0) - \frac{1}{2} S(0)^{2(\beta-1)} \int_0^t \sigma(u)^2 du \\ & + S(0)^{\beta-1} \frac{\rho}{\alpha} (\sigma(t) - \sigma(0)) \\ & + S(0)^{\beta-1} \sqrt{1-\rho^2} \int_0^t \sigma(u) dW_S(u)\end{aligned}$$

We dub this the Log-Euler+ scheme, which has to be paired with a scheme for the integrated variance.

# SABR schemes (9)

Finally, the Ninomiya-Victoir “splitting” scheme with drift due to Bayer, Friz and Loeffen [2013] is also considered for the SABR model:

- ▶ The NV scheme with drift is a slight adaptation of the original NV scheme, and opens up closed-form schemes for a wide range of classes, also for SABR;
- ▶ For  $\frac{1}{2} < \beta \leq 1$  their scheme has second-order weak convergence;
- ▶ Bayer, Friz and Loeffen [2013] mention that  $\beta = \frac{1}{2}$  is a problematic case (the boundary at zero has to be dealt with separately).

## *Part II – Integrated variance*

# *Integrated variance*

The integrated variance process:

$$\int_0^t \sigma(u)^2 du \mid \sigma(0), \sigma(t)$$

is clearly related to the problem of pricing a continuously sampled arithmetic Asian option, see Geman and Yor [1993]. The difference lies in the additional conditioning on the end point.

Matsumoto and Yor [2005] provide a double Laplace transform of the density, but the involved integrals are extremely challenging, more so than in the Asian case (Fu, Madan and Wang [1999], Shaw [1998, 1999]).

# *Integrated variance (2)*

Chen, Oosterlee and Van der Weide [2012] use a small disturbance expansion to arrive at an approximate first moment and variance, conditional upon the already simulated  $\sigma(t)$ . The idea is to write:

$$\sigma^{(\varepsilon)}(t) = \sigma(0) + \varepsilon \tilde{\alpha} \int_0^t \sigma^{(\varepsilon)}(u) dW_\sigma(u)$$

where  $\alpha = \varepsilon \tilde{\alpha}$ , and expand around  $\varepsilon = 0$ . One can then derive the first and second moment of this expansion.

Finally, a lognormal distribution is fitted to these two parameters.

# *Integrated variance (3)*

As we will see, Chen et al.'s algorithm is only suited for small values of  $\alpha^2 t$ . For short to medium time steps, one can therefore question whether simpler schemes are not more effective. For example:

- ▶ As Andersen [2008], a central discretisation / trapezoidal rule:

$$\int_0^t \sigma(s)^2 ds \approx \left( \frac{1}{2} \sigma(0)^2 + \frac{1}{2} \sigma(t)^2 \right) \cdot t$$

- ▶ Simulating additional intermediate volatilities:

$$\int_0^t \sigma(s)^2 ds \approx \sum_{i=0}^N w_i \sigma(t_i)^2$$

# *Integrated variance (4)*

Since the moments can be calculated analytically (Kennedy, Mitra and Pham [2012]), moment matching can also be performed. For example, one finds:

$$\mathbb{E}[\sigma(u)^2 | \sigma(t)] = \sigma(0)^2 \left( \frac{\sigma(t)^2}{\sigma(0)^2} \right)^{\frac{u}{t}} \exp\left(\alpha^2 \left(1 - \frac{u}{t}\right)(t + 2u)\right)$$

so that we can approximate:

$$\mathbb{E}\left[\int_0^t \sigma(u)^2 du | \sigma(t)\right] = \int_0^t \mathbb{E}[\sigma(u)^2 | \sigma(t)] du$$

with a small quadrature. One can proceed similarly for the second moment.

# *Integrated variance (5)*

Techniques that have been used with success in Asian option pricing, see e.g. Lord [2006], can also be used, and are probably more suited in a long-stepped simulation context. From Jensen's inequality we have:

$$V(t) = \int_0^t \sigma(s)^2 ds \geq t \exp\left(\frac{1}{t} \int_0^t \ln \sigma(s)^2 ds\right) = G(t)$$

We can therefore reduce the variance by additional conditioning on the geometric integrated variance.

Since the latter, conditional on  $\sigma(0)$  and  $\sigma(t)$ , is still lognormally distributed, this is definitely feasible.

# *Integrated variance (6)*

The suggested algorithm becomes:

- ▶ Simulate  $\sigma(t) | \sigma(0)$
- ▶ Simulate  $G(t) | \sigma(t), \sigma(0)$
- ▶ Fit a distribution to  $V(t) | G(t), \sigma(t), \sigma(0)$ , which takes into account that  $V(t) \geq G(t)$

For small to medium-sized steps the following could be sufficiently accurate (we dub this Curran1M+):

$$V(t) | G(t), \sigma(t), \sigma(0) \sim \mathbb{E}[V(t) | G(t)]$$

# *Integrated variance (7)*

If more accuracy is required, one could consider the Curran2M+ approximation of Lord [2006]:

$$V(t) | G(t), \sigma(t), \sigma(0) \sim G(t) + \exp(\mu + \sigma \cdot Z)$$

where  $\mu$  and  $\sigma$  are fitted to the first two moments of the conditional distribution.

The Curran2M+ approach, in a Black-Scholes setting, can approximate a 30y Asian option, with yearly averaging, and a 25% implied volatility, within basis point accuracy, so it certainly should be appropriate for this application.

# *Integrated variance (8)*

In the following example we compare:

- ▶ **Chen et al.** [2012]’s scheme
- ▶ An **Euler** scheme, that is  $\int_0^t \sigma(s)^2 ds \approx \sigma(0)^2 \cdot t$
- ▶ **Central** scheme / trapezoidal rule
- ▶ **Levy** – (cf. Levy [1992]) 2-moment matching to a lognormal random variable, using a 3-point Gauss-Legendre quadrature to approximate the moments
- ▶ **Curran1M+** and **Curran2M+** schemes
- ▶ **Ninomiya-Victoir** scheme with drift

# *Integrated variance (9)*

In order to avoid any discretisation error, we focus on the  $\beta = 0$  case, where  $S(t)$  can be found as:

$$S(t) = S(0) + \frac{\rho}{\alpha} (\sigma(t) - \sigma(0)) + \sqrt{1 - \rho^2} \int_0^t \sigma(u) dZ(u)$$

Here  $Z$  is a Brownian motion that is independent of the Brownian motion driving the volatility.

In addition, we can use Willard's [1997] conditioning technique to reduce the variance. Finally, the reference price is known from Korn and Tang [2013], who derive a closed-form expression for European prices in the normal SABR model.

# *Integrated variance (10)*

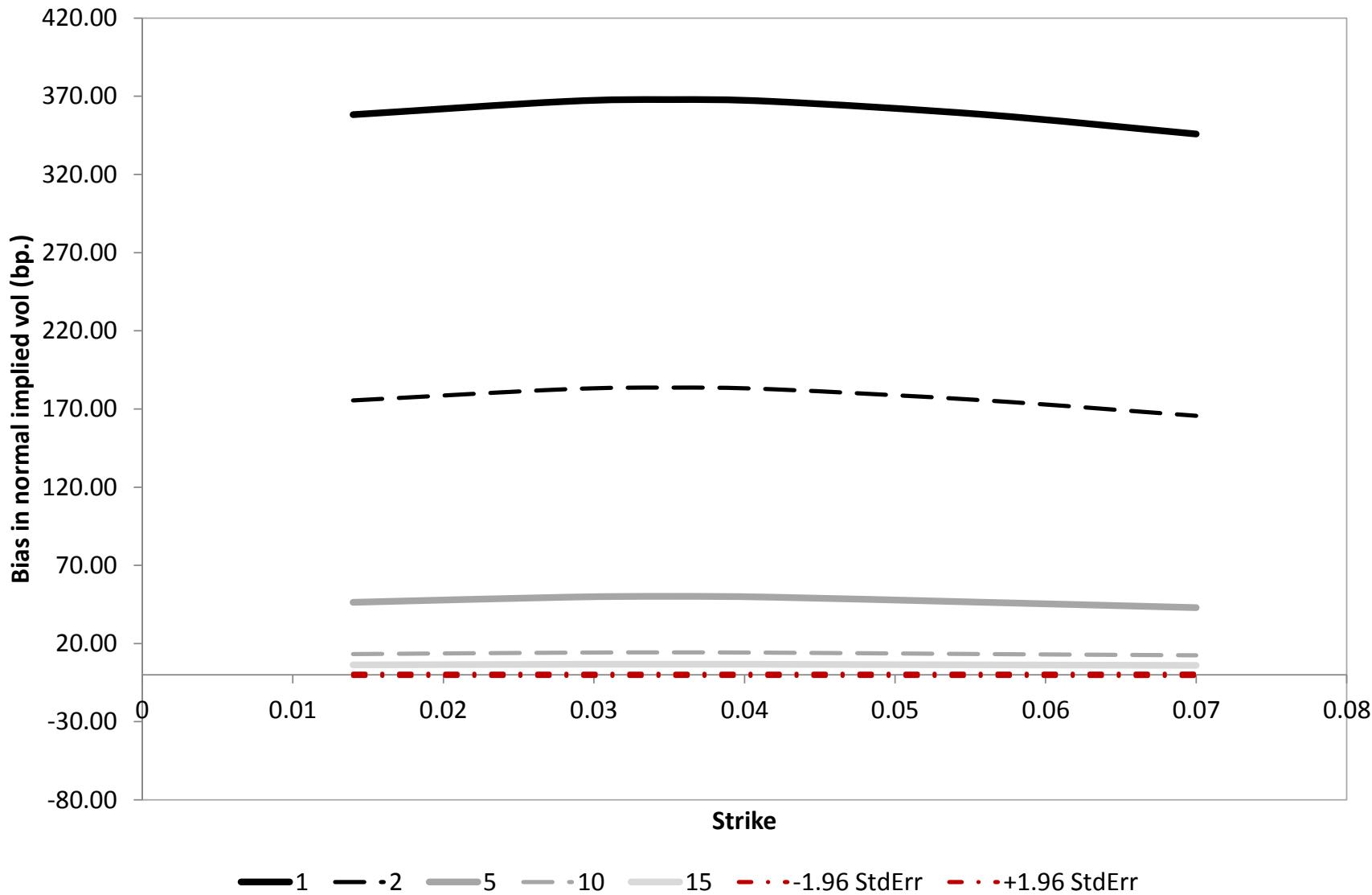
The parameter set we have investigated here is Test Case II from Korn and Tang [2013]:

$\sigma(0)$	$\alpha$	$\beta$	$\rho$	$S(0)$
1%	50%	0	0%	3.5%

where we looked at a 30y European call option.

To generate the simulation results (also all others in this study), 100 million paths were used.

# *Integrated variance – Chen et al.*



## *Integrated variance – Chen et al. (2)*

The approximation for the first moment resulting from Chen et al. [2012] is a good approximation. The variance however, only is accurate when  $\alpha^2 t$  is very small, which rules out the usage with large time steps.

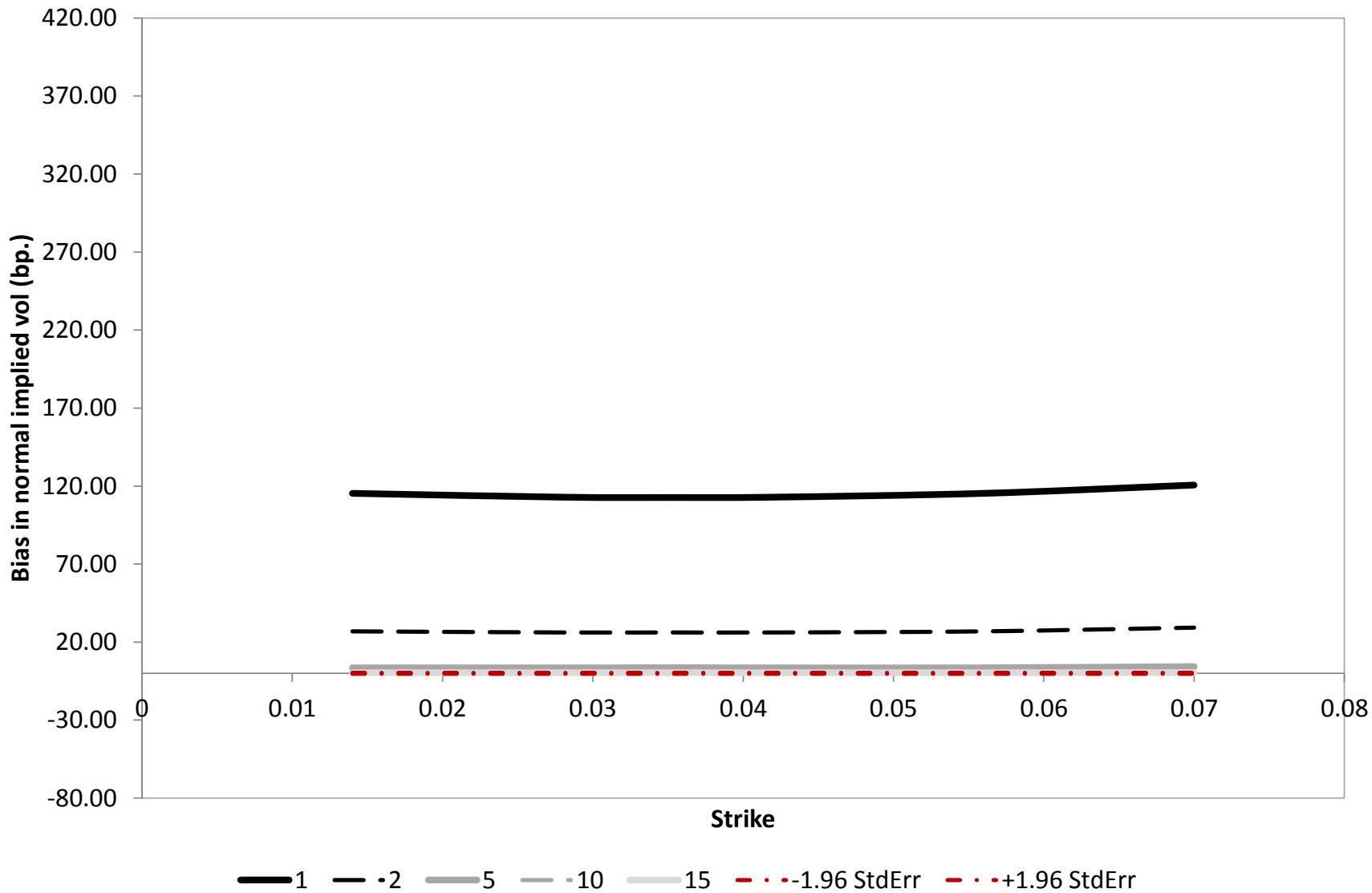
Chen et al.'s variance approximation yields:

$$\sigma_{\text{ChenEtAl}}^2(t) = \frac{1}{3} \alpha^2 \sigma(0)^4 t^3$$

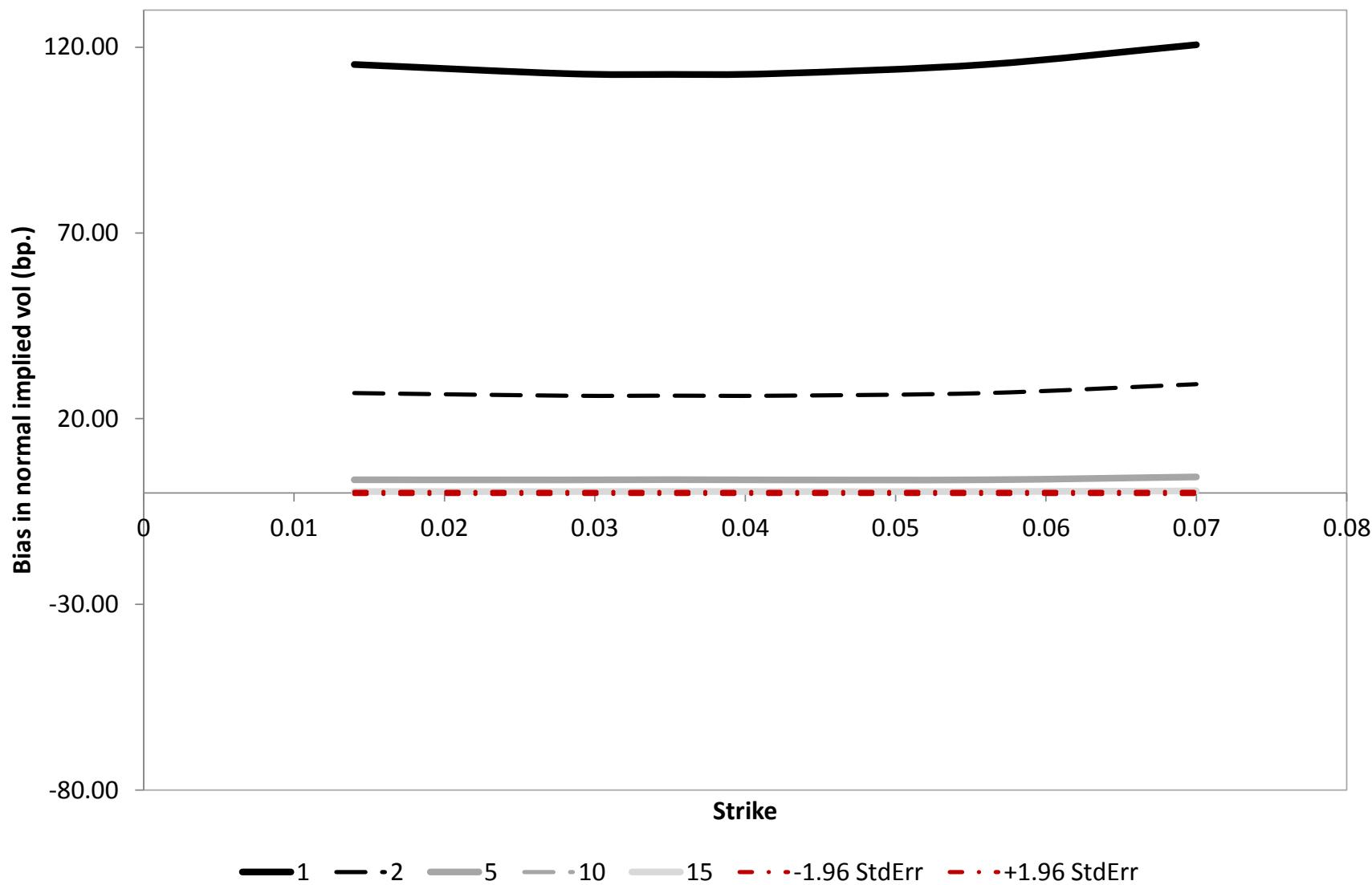
whereas the variance of the geometric variance equals:

$$(t\sigma(0)\sigma(t))^2 (e^{\frac{1}{3}\alpha^2 t} - 1) \geq \frac{1}{3} \alpha^2 \sigma(0)^2 \sigma(t)^2 t^3$$

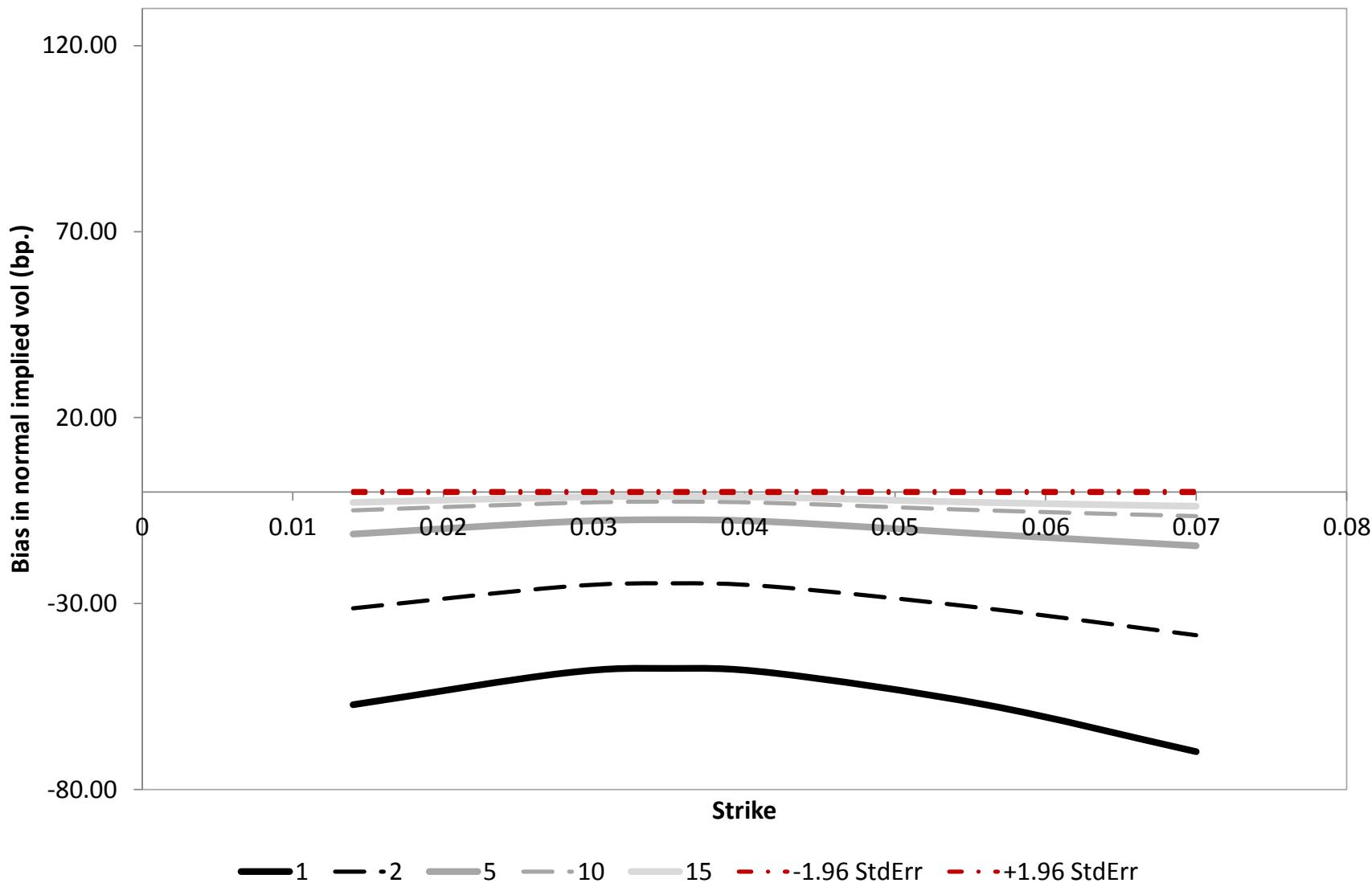
# *Integrated variance – NV with drift*



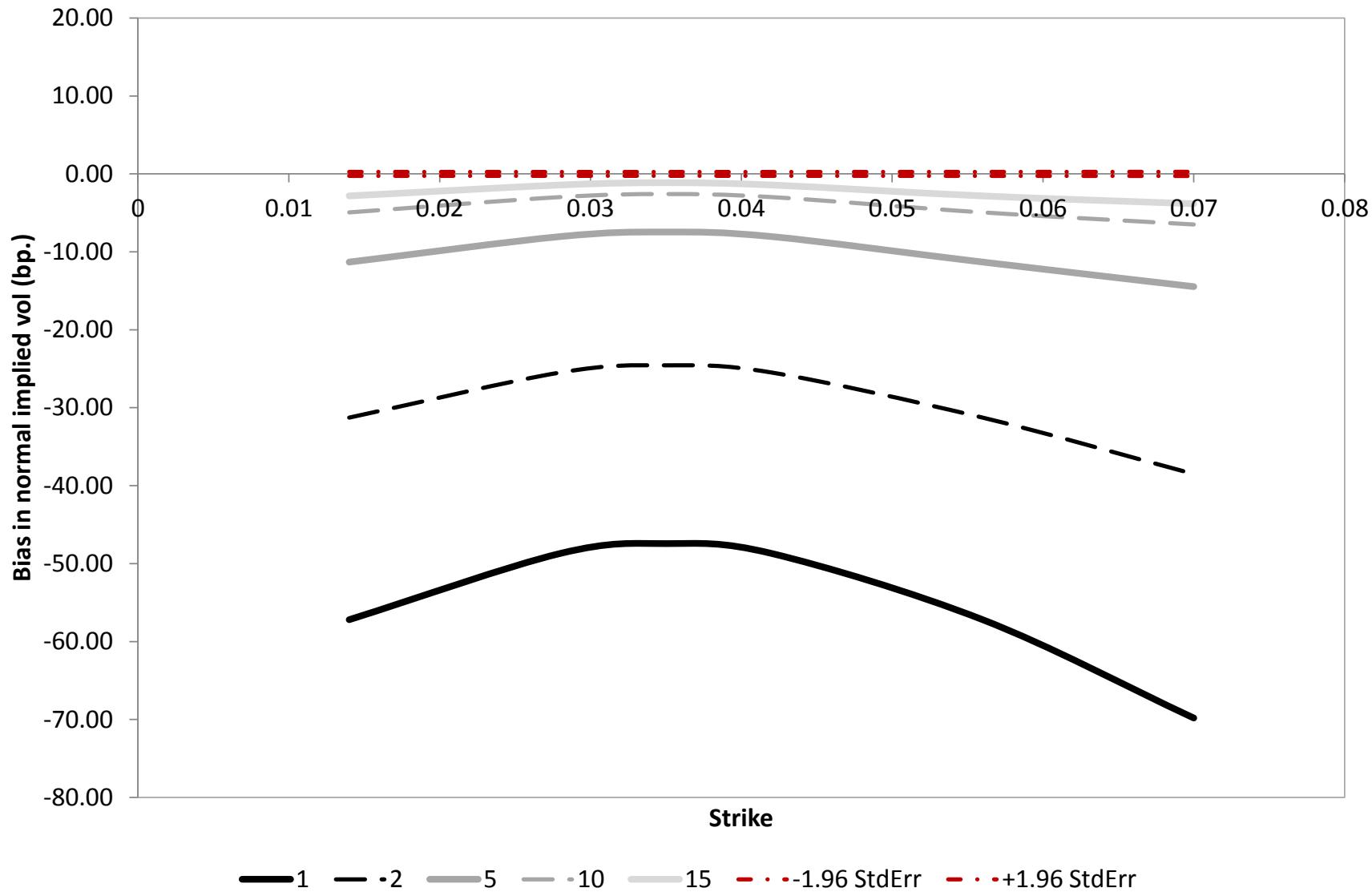
# *Integrated variance – NV with drift (zoom)*



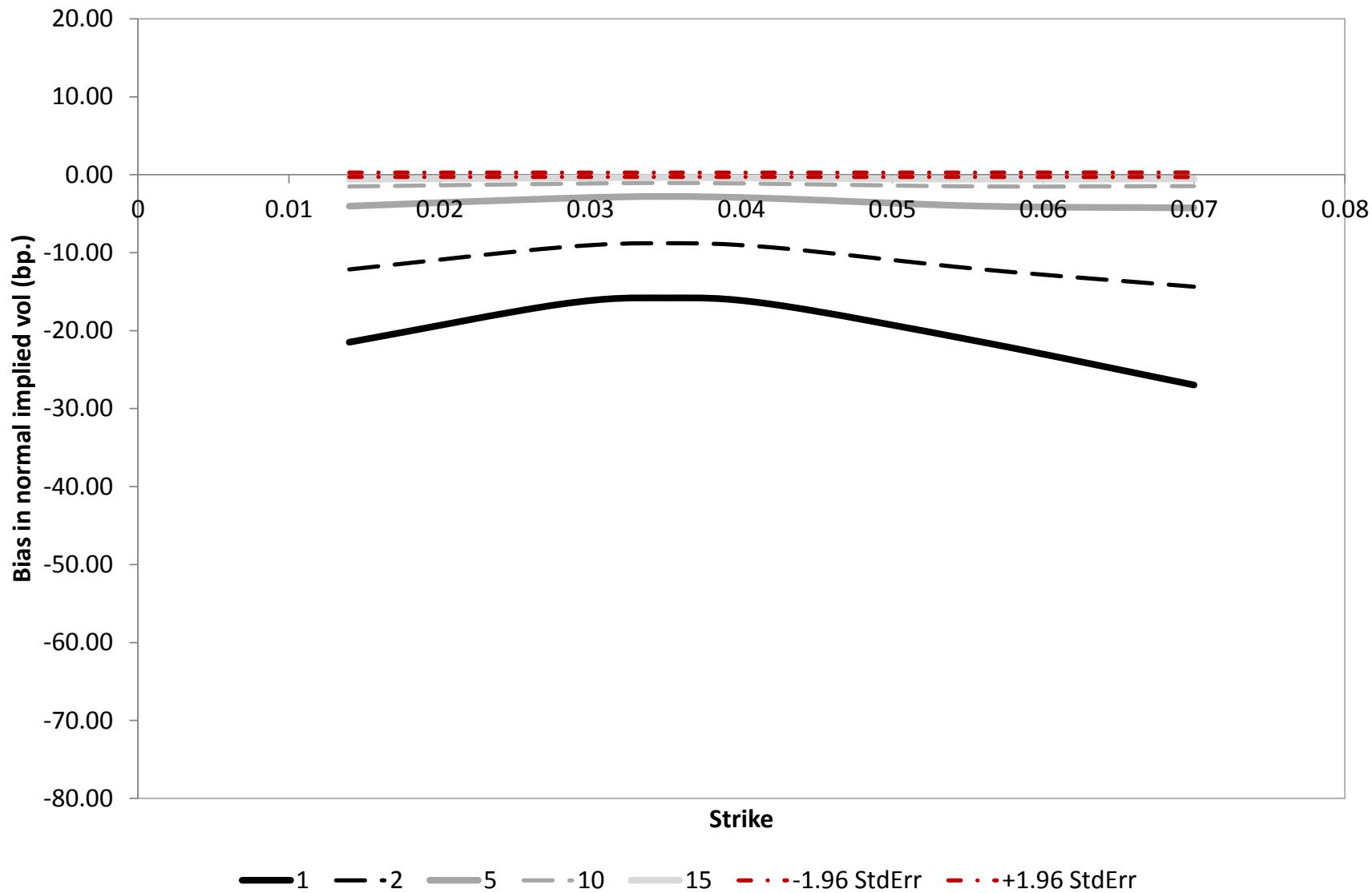
# *Integrated variance – Euler*



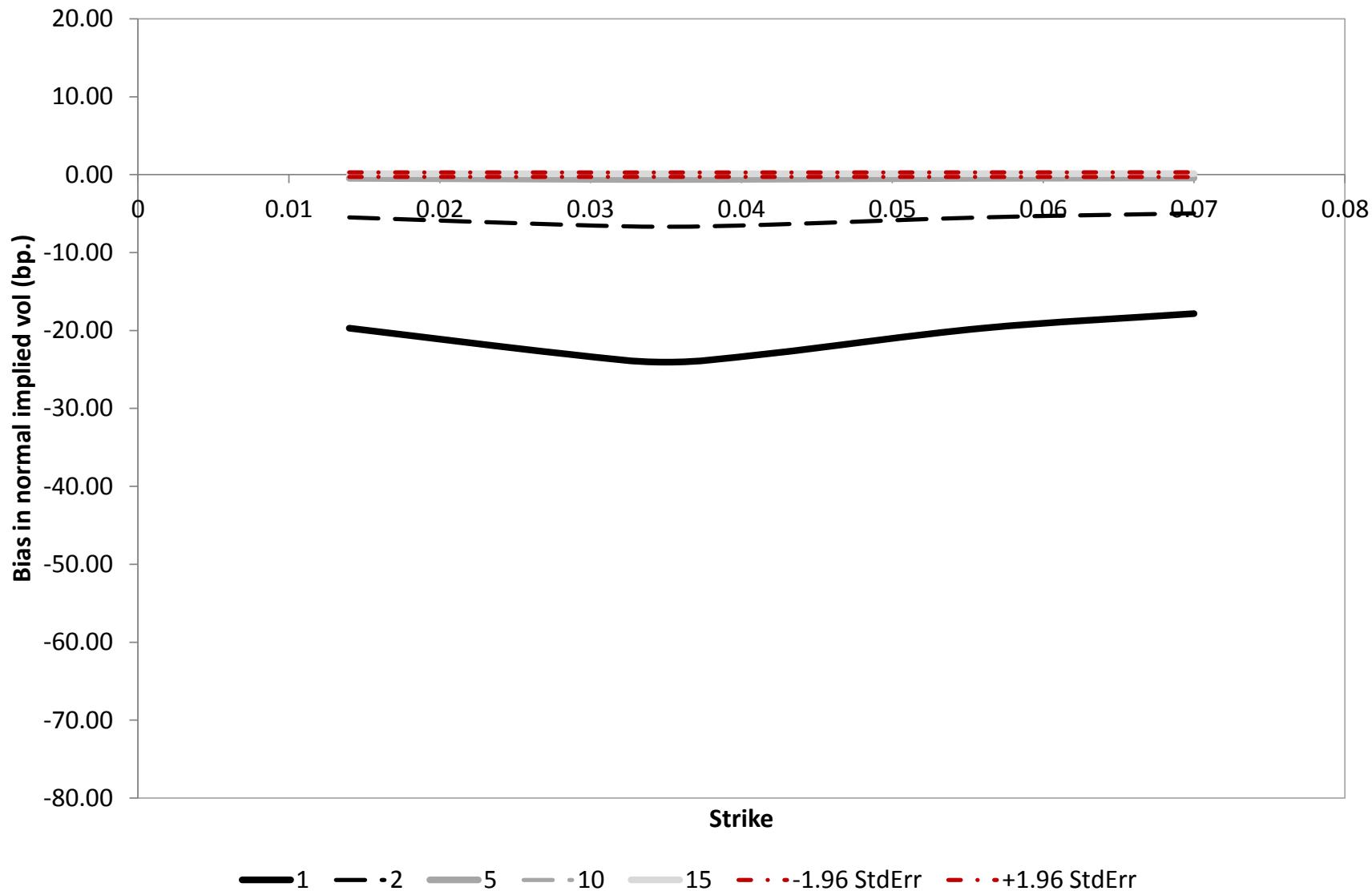
# *Integrated variance – Euler (zoom)*



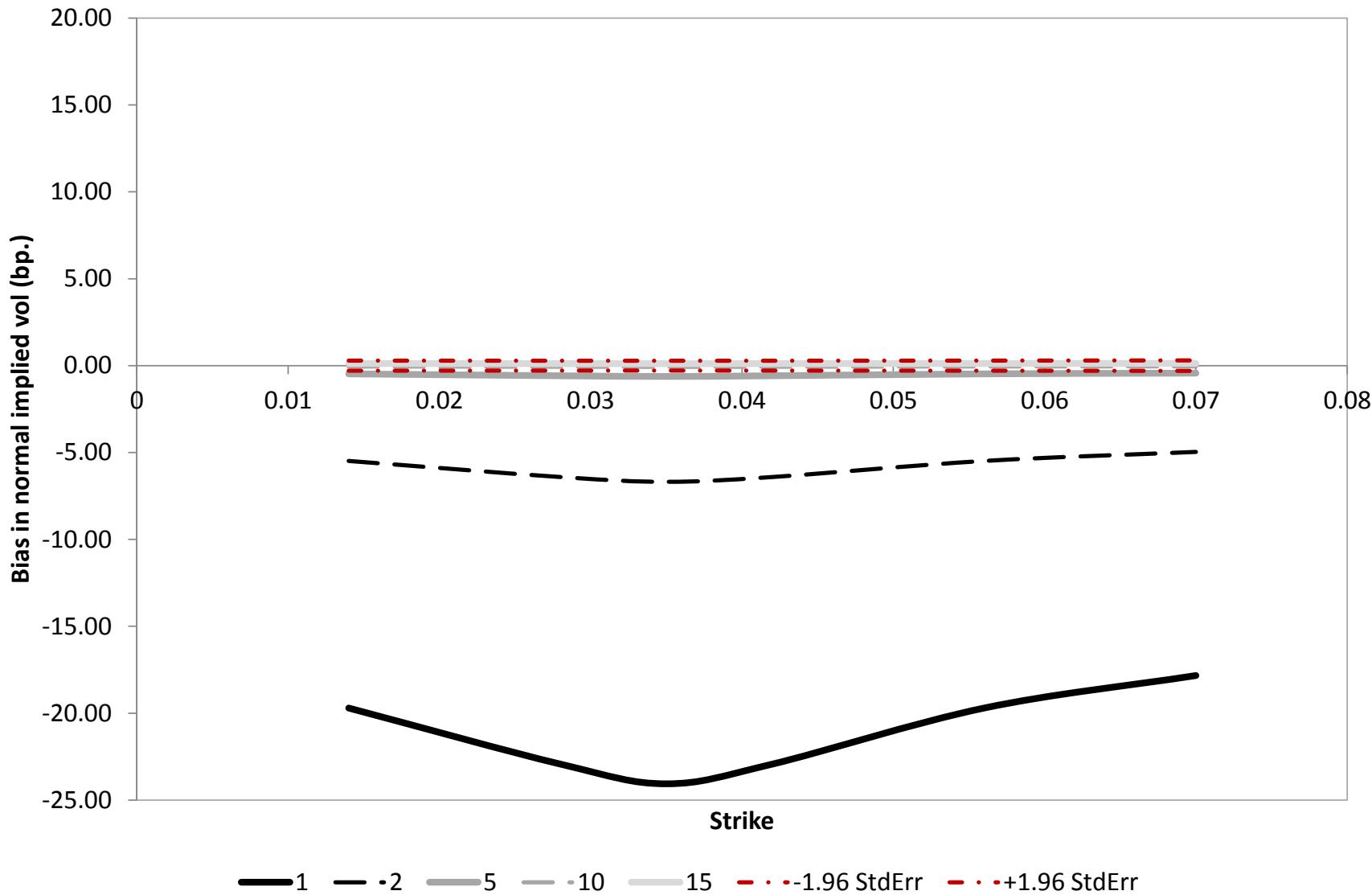
# *Integrated variance – Central*



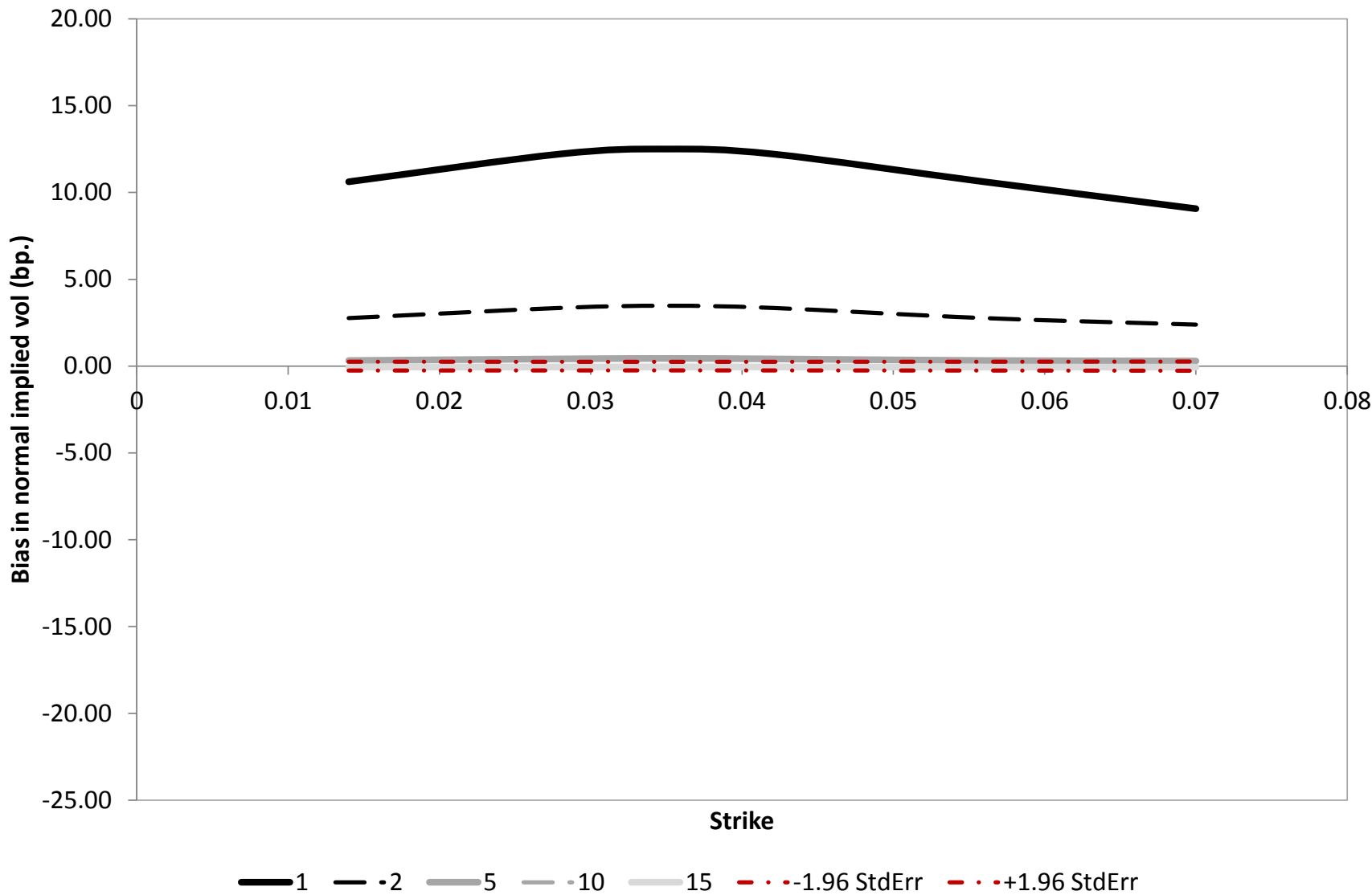
# *Integrated variance – Levy*



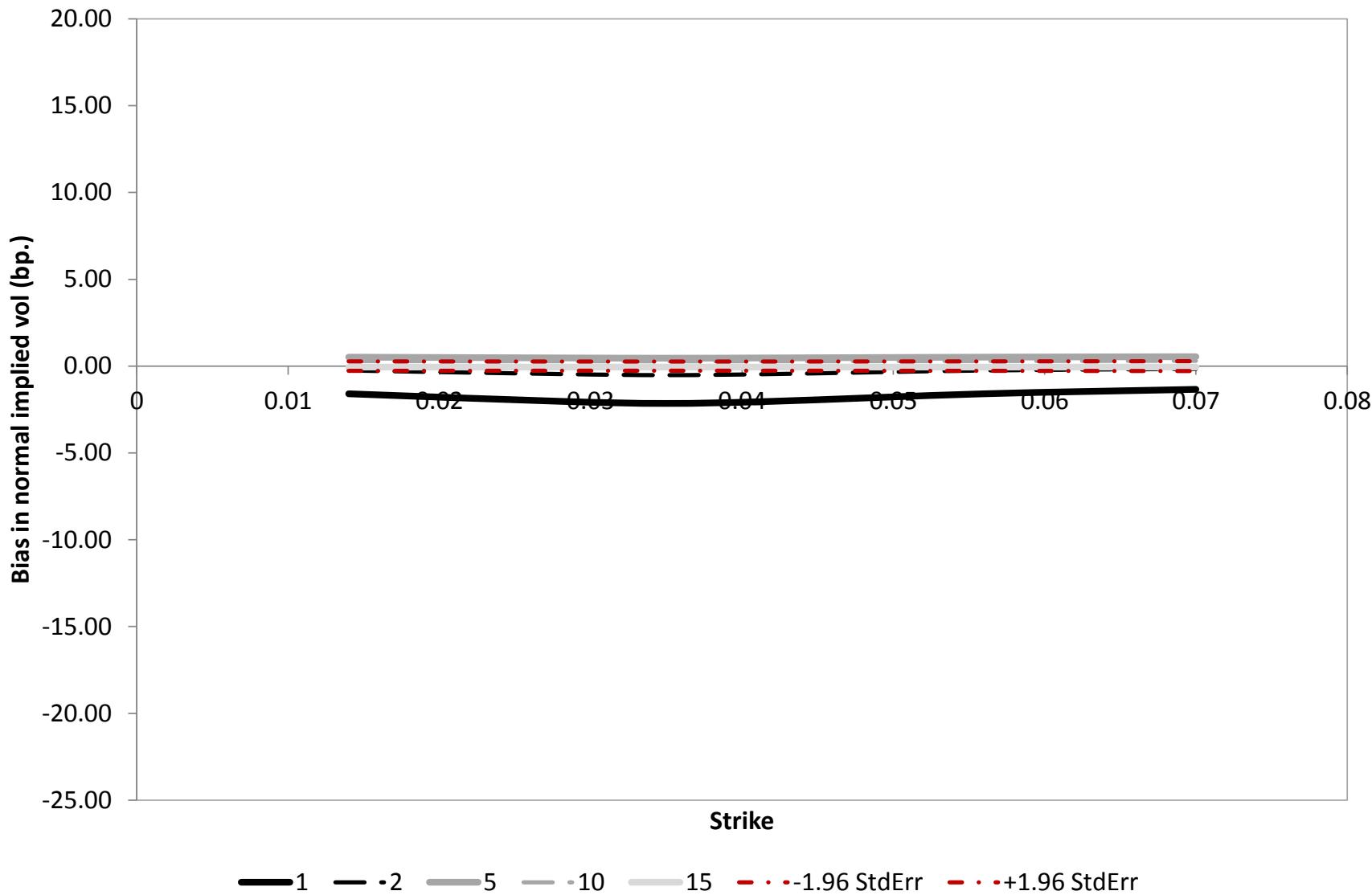
# *Integrated variance – Levy (zoom)*



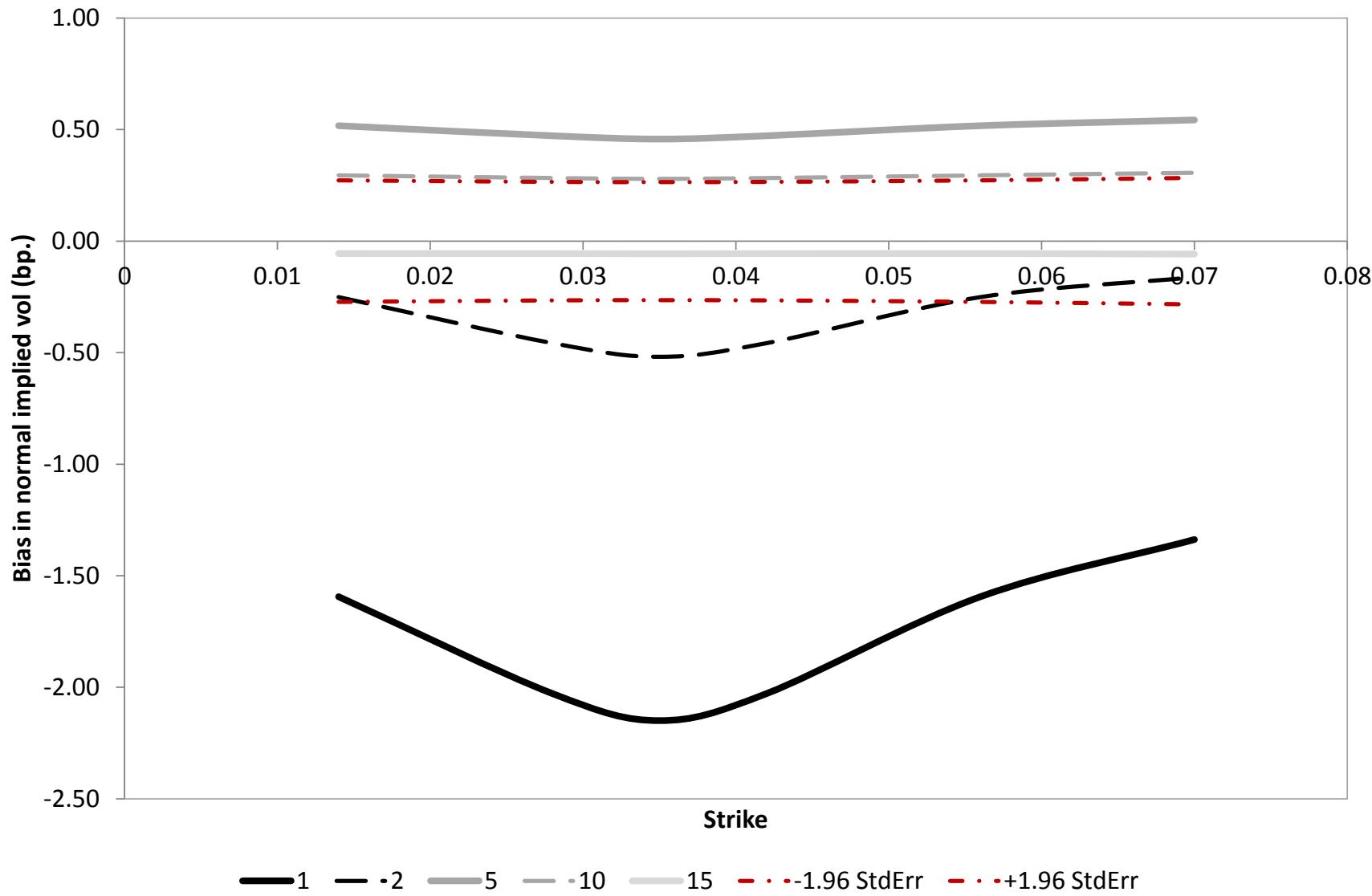
# *Integrated variance – Curran 1M+*



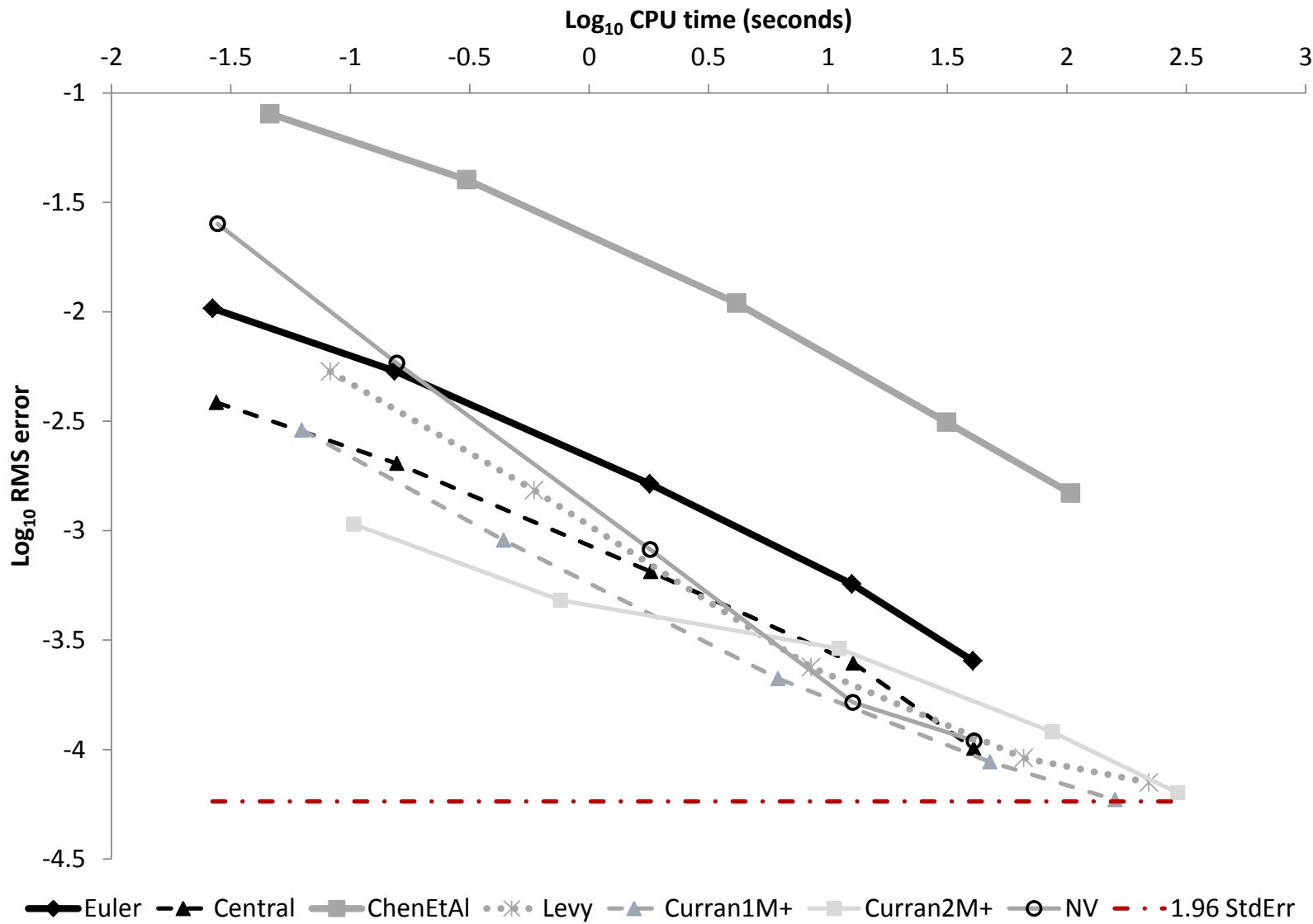
# *Integrated variance – Curran2M+*



# *Integrated variance – Curran2M+ (zoom)*



# *Integrated variance – RMS error*



# *Part III – Moment-matching the asset*

# *Moment matching the asset*

Using known results on the squared Bessel process one can show that the cdf of the asset, conditional on the volatility and integrated variance, can be approximated by:

$$P\left(S(t) \leq x \mid S(0) > 0, \int_0^t \sigma(u)^2 du, \sigma(t)\right) = 1 - \chi^2(a, b, c)$$

where the parameters  $a$ ,  $b$  and  $c$  are determined by:

$$a = \frac{\tilde{X}(0)^2}{v(t)} \quad b = 2 - \delta \quad c = \frac{x^{2(1-\beta)}}{(1-\beta)^2 v(t)}$$

Here  $\tilde{X}(0)$ ,  $\delta$  and  $v(t)$  are as defined before.

# **Moment matching the asset (2)**

Chen, Oosterlee and Van der Weide [2012] recently argued that:

$$1 - \chi^2(a, b, c) = \chi^2(c, 2 - b, a) + P(\inf \{ u | S(u) = 0 \} < t)$$

Based on this they suggest to simulate the asset, conditional on volatility and integrated variance, as:

- ▶ If  $S(0) \gg 0$ : Quadratic Gaussian, moment-matched to first two moments of  $\chi^2(c, 2-b, a)$ ;
- ▶ For small  $S(0)$ ,  $1 - \chi^2(a, b, c)$  is inverted directly;
- ▶ The absorption probability is handled in all cases.

# *Moment matching the asset (3)*

Since the inversion step is expensive, and would require a 2D cache in order to be sped up, we turn to the moments of the squared Bessel process. For:

$$dY(t) = 2\sqrt{Y(t)} dW(t) + \delta dt$$

one can show the  $a^{\text{th}}$  moment ( $a > -1$ ,  $\delta < 2$ ) equals:

$$\mathbb{E}[Y(t)^a | Y(0) = y] = (2t)^{-b} y^{1-\frac{1}{2}\delta} \frac{\Gamma(1+a)}{\Gamma(2-\frac{1}{2}\delta)} \cdot {}_1F_1(b, 2 - \frac{1}{2}\delta, -\frac{y}{2t})$$

where  $b = 1 - a - \frac{1}{2}\delta$ .

Contrary to Chen et al., the approximate moments can be calculated and moment matching can be employed.

# *Moment matching the asset (4)*

The “detail” we have glossed over is the calculation of the confluent hypergeometric function. Not many numerical libraries have an implementation hereof (though NAG has since Mark 24, and GSL does too).

Since the first two parameters are fixed per run, it seems most efficient to solve the defining ODE, and cache the outcomes. For  $y(z) = {}_1F_1(a, b, z)$ , it is:

$$z \frac{d^2y}{dz^2} + \frac{dy}{dz} (b - z) - ay(z) = 0$$

where  $y(0) = 1$  and  $y'(0) = a/b$ .

# *Moment matching the asset (5)*

In Andersen's [2008] QE scheme for Heston, for low values of the variance, a mixture distribution of an exponential and zero are used for moment matching.

Essentially such a mixture distribution has the form:

$$G(x) = p + (1 - p)F(x)$$

where  $G(x)$  is the approximating cdf, and  $F(x)$  is an analytically tractable cdf that we want to use in the mixture (exponential cdf in Andersen's scheme).

# *Moment matching the asset (6)*

We could certainly use such a scheme here. The difference is that we do not want to imply  $p$  in order to fit the first two moments, but we can match it exactly to the absorption probability, which can be calculated exactly for a squared Bessel process.

The first two moments of  $G$  satisfy:

$$m_G = (1 - p)m_F$$

$$s_G^2 + m_G^2 = (1 - p)(s_F^2 + m_F^2)$$

with  $m$  the first moment, and  $s$  the standard deviation.

# *Moment matching the asset (7)*

Rearranging these equations in terms of first moments and the squared coefficient of variation  $\psi = s^2 / m^2$ :

$$m_F^2 = \frac{m_G^2}{(1-p)^2}$$

$$1 + \psi_F = (1 + \psi_G)(1 - p)$$

Simulating from such a distribution is straightforward:

$$G^{-1}(U) = \begin{cases} 0 & U < p \\ F^{-1}\left(\frac{U-p}{1-p}\right) & U \geq p \end{cases}$$

# **Moment matching the asset (8)**

For the choice of distribution, the distribution  $F$  can, as in Andersen's scheme, be chosen to be quadratic Gaussian. It is known that its squared coefficient of variation can only lie in  $[0, 2]$ .

Therefore, like in Andersen's scheme, we can choose a threshold value  $\psi_{\text{threshold}} \in [0, 2]$  and choose  $F$  to be:

- ▶ A quadratic Gaussian cdf if  $\psi \leq \psi_{\text{threshold}}$ ;
- ▶ Another cdf (e.g. lognormal) if  $\psi > \psi_{\text{threshold}}$ .

# *Moment matching the asset (9)*

A nice alternative for the high variation distribution that is in line with Andersen's QE scheme, is the hyperexponential distribution  $H_2^b$  with balanced means. Its cdf satisfies:

$$pF_{\lambda_1}(x) + (1-p)F_{\lambda_2}(x)$$

where  $F_\lambda$  is the cdf of an exponential distribution. The balanced means assumption entails that:

$$p\lambda_1^{-1} = (1-p)\lambda_2^{-1}$$

see e.g. Tijms [2003] or Adan and Resing [2002].

# **Moment matching the asset (10)**

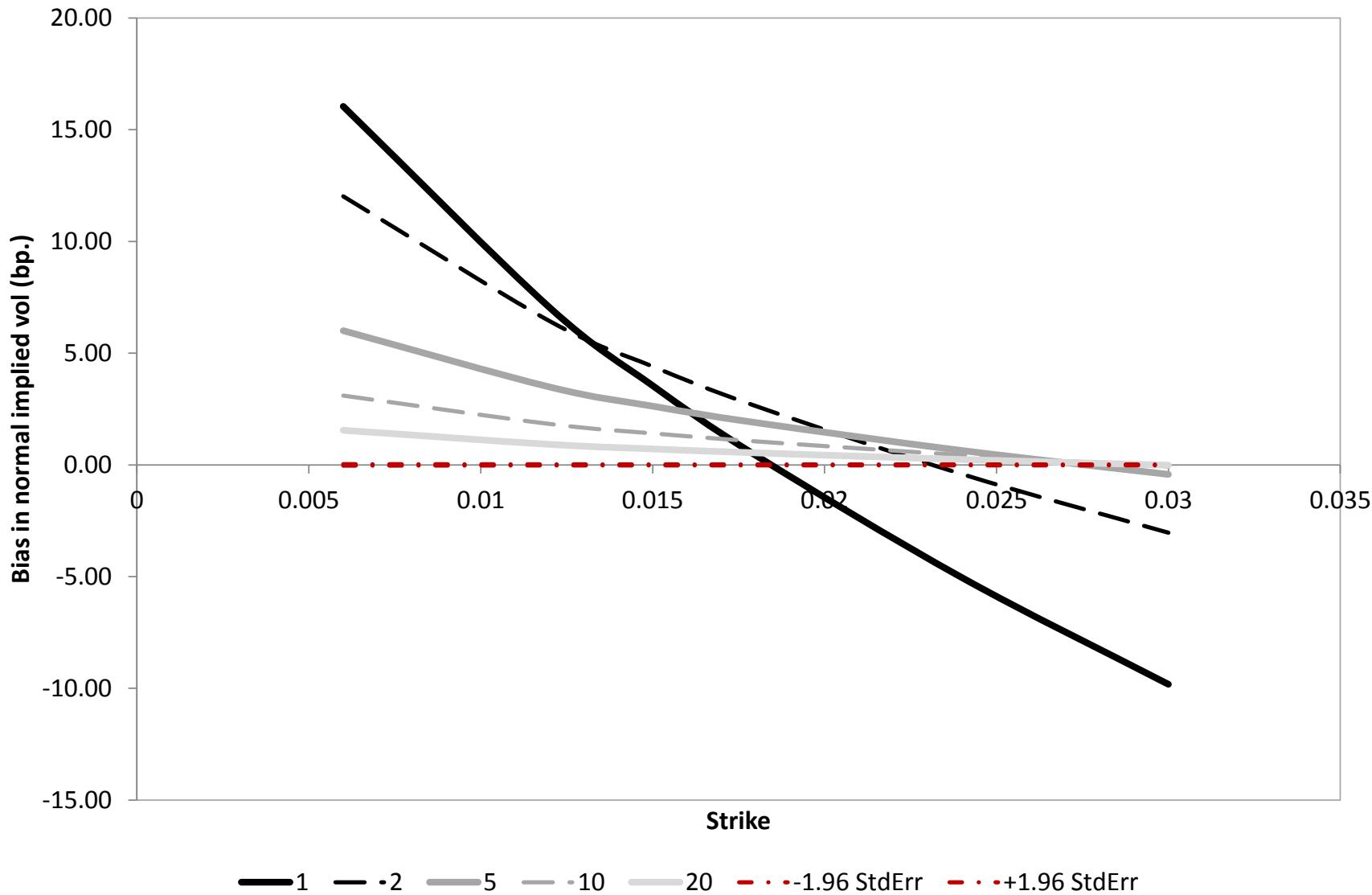
To test the accuracy of the moment-matching schemes, we focus on an example where there is no influence from the approximations made in the integrated variances process.

Therefore we focus on a 10y call in a CEV example:

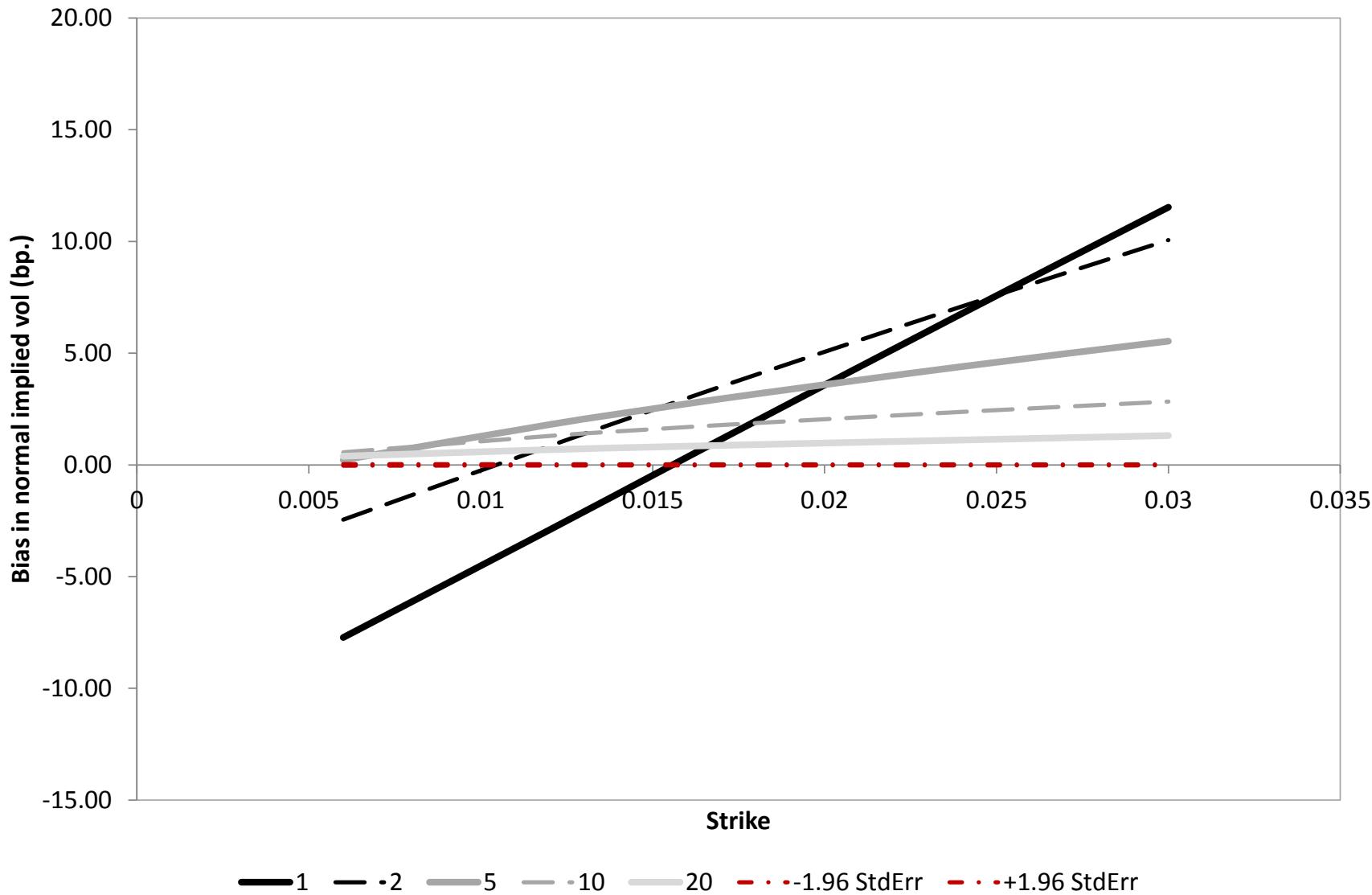
$\sigma$	$\beta$	$S(0)$
5%	0.5	1.5%

We compare two moment-matching schemes (Quadratic Gaussian, lognormal) to Euler, Log-Euler and Ninomiya-Victoir discretisations, as well as to a Displaced Diffusion scheme (Kennedy et al. [2012]).

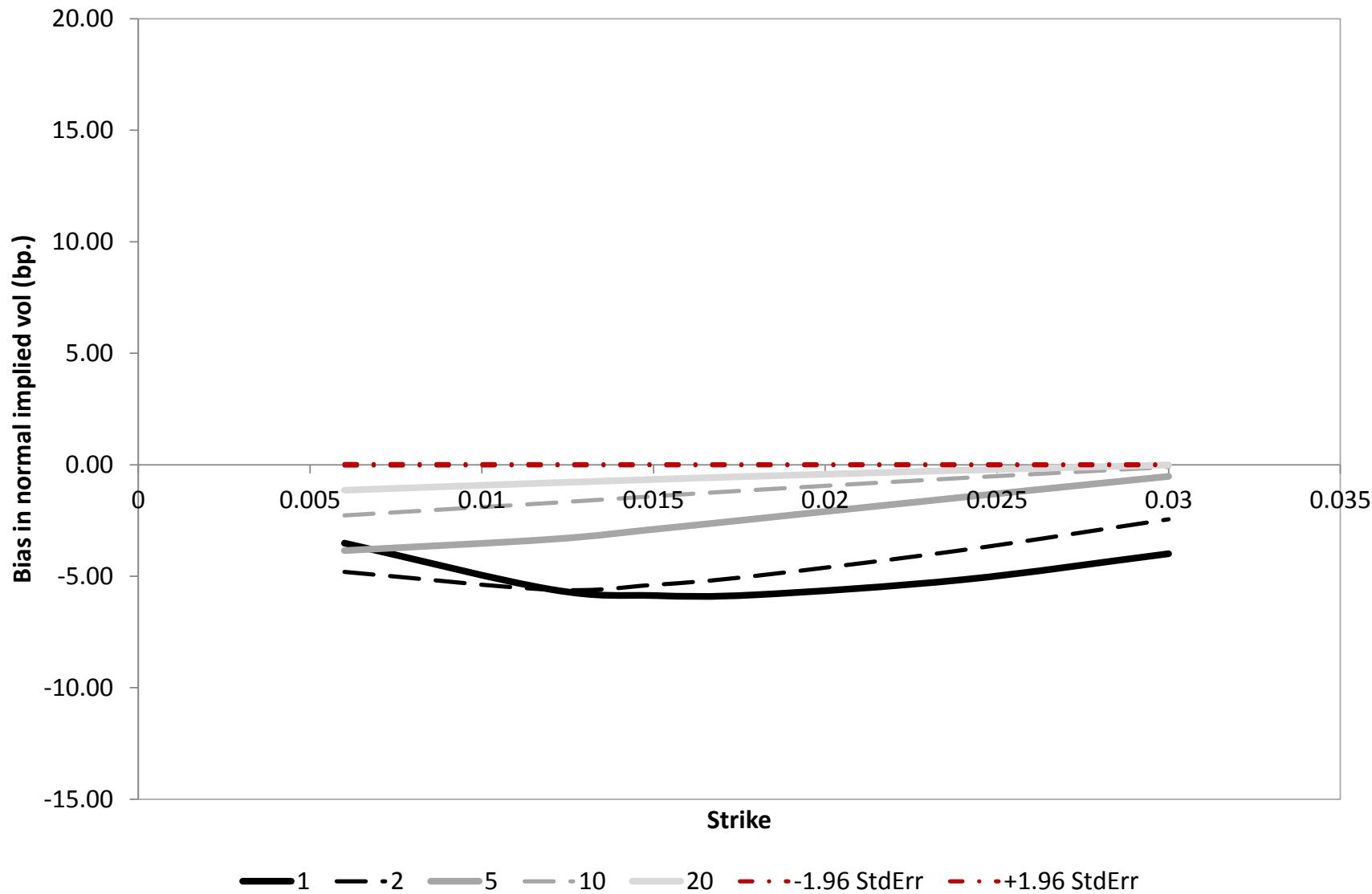
# Moment matching - Euler



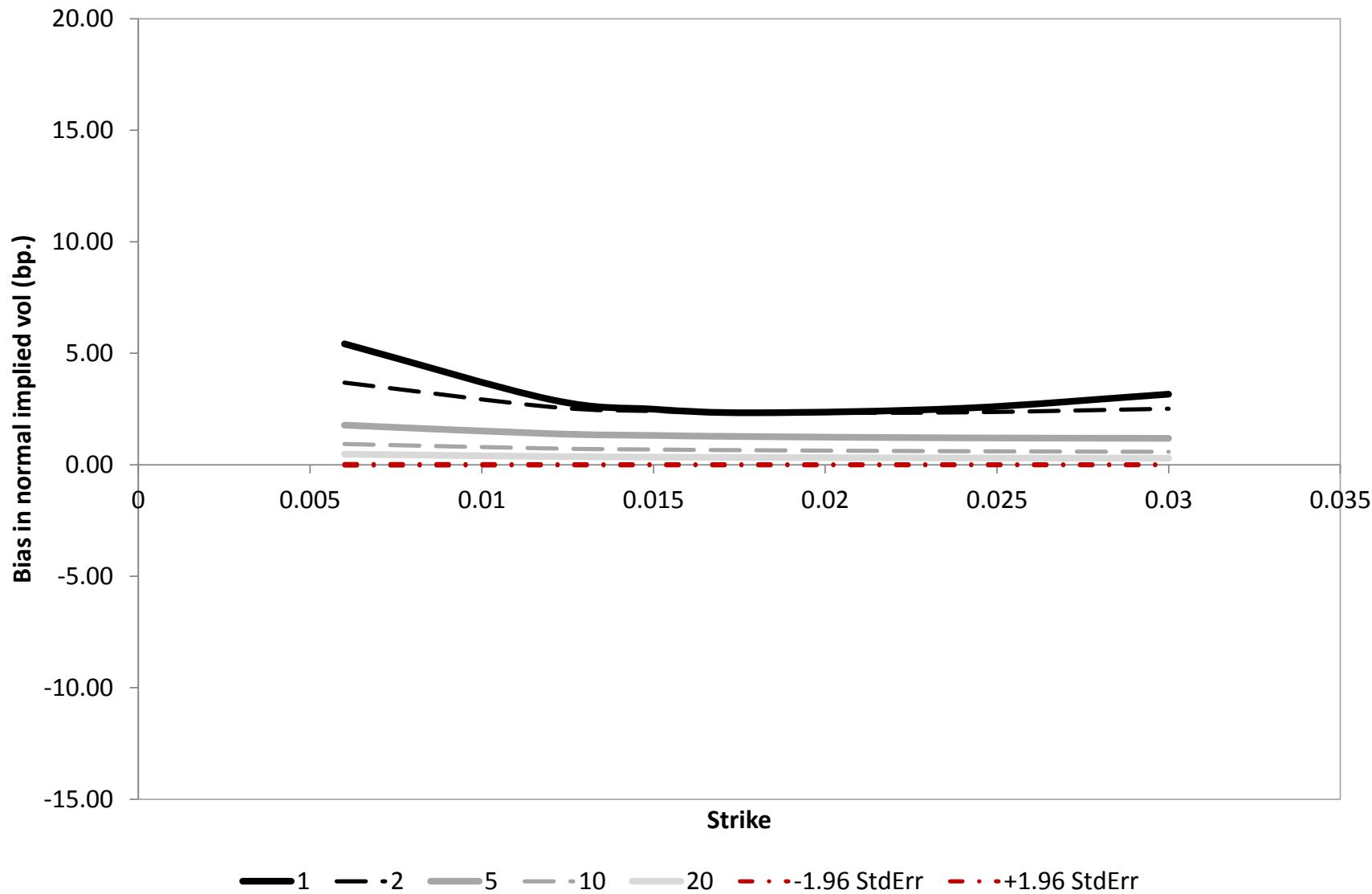
# Moment matching - Log-Euler



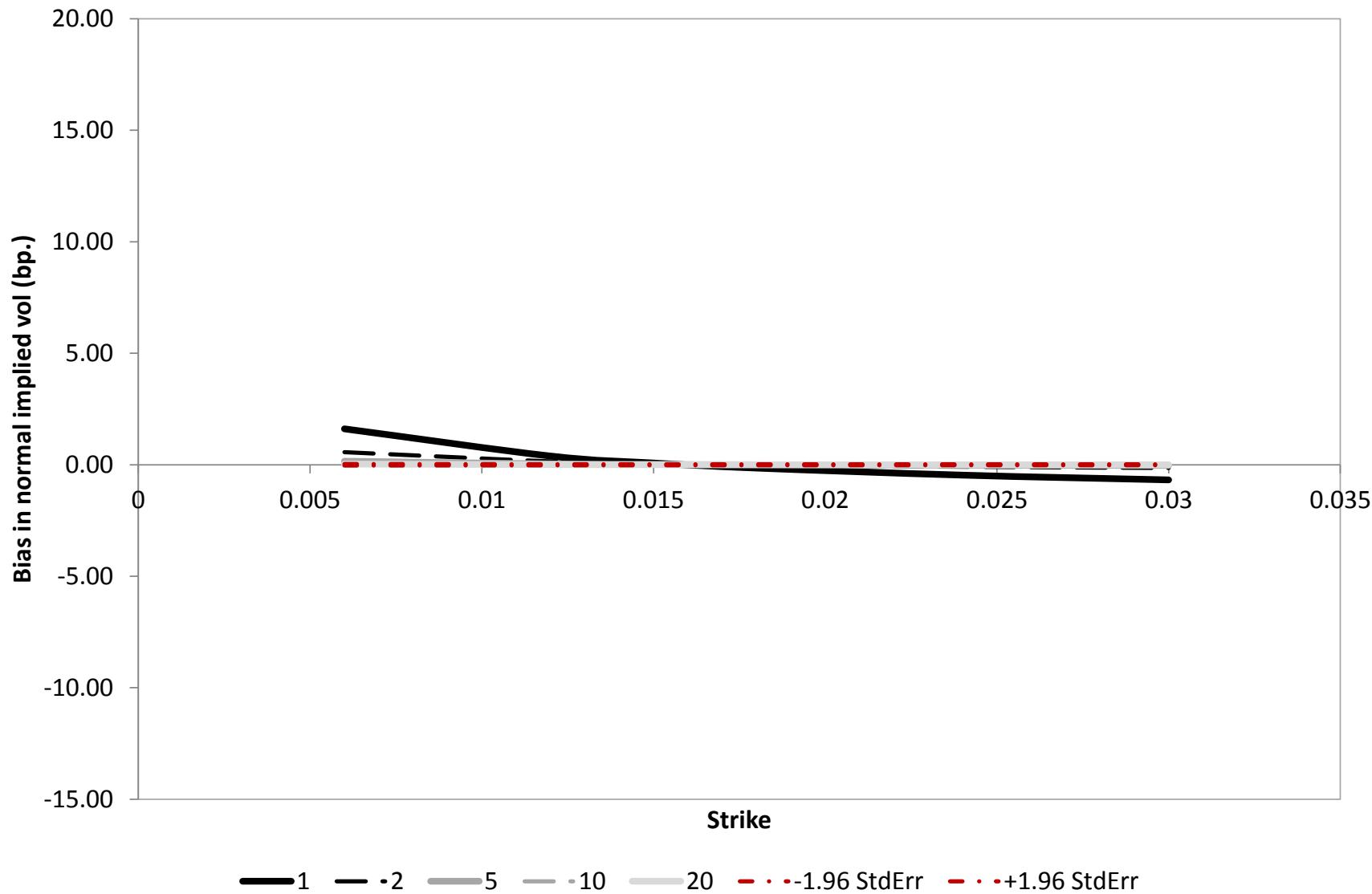
# Moment matching - Lognormal



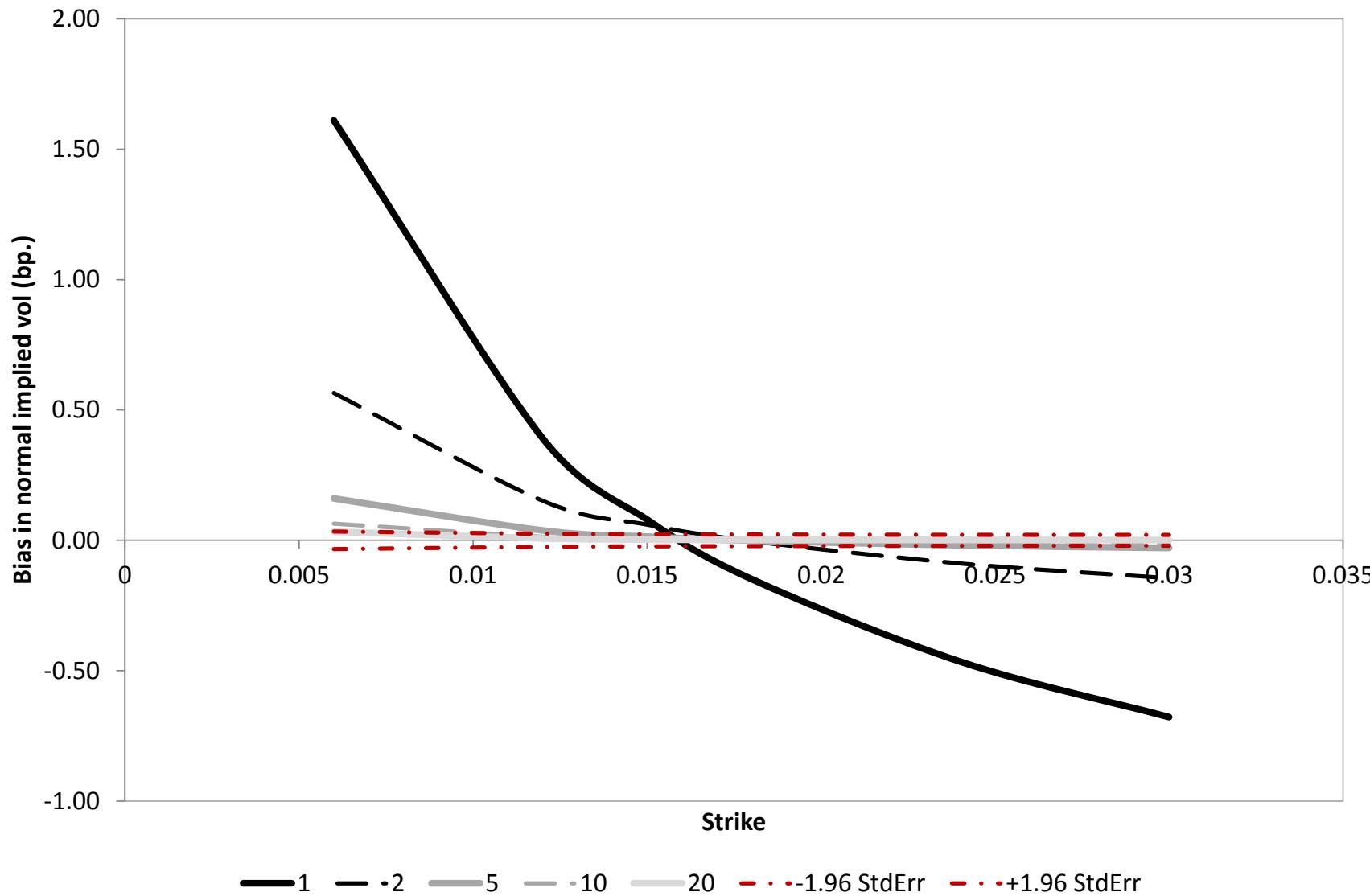
# *Moment matching – Displaced Diffusion*



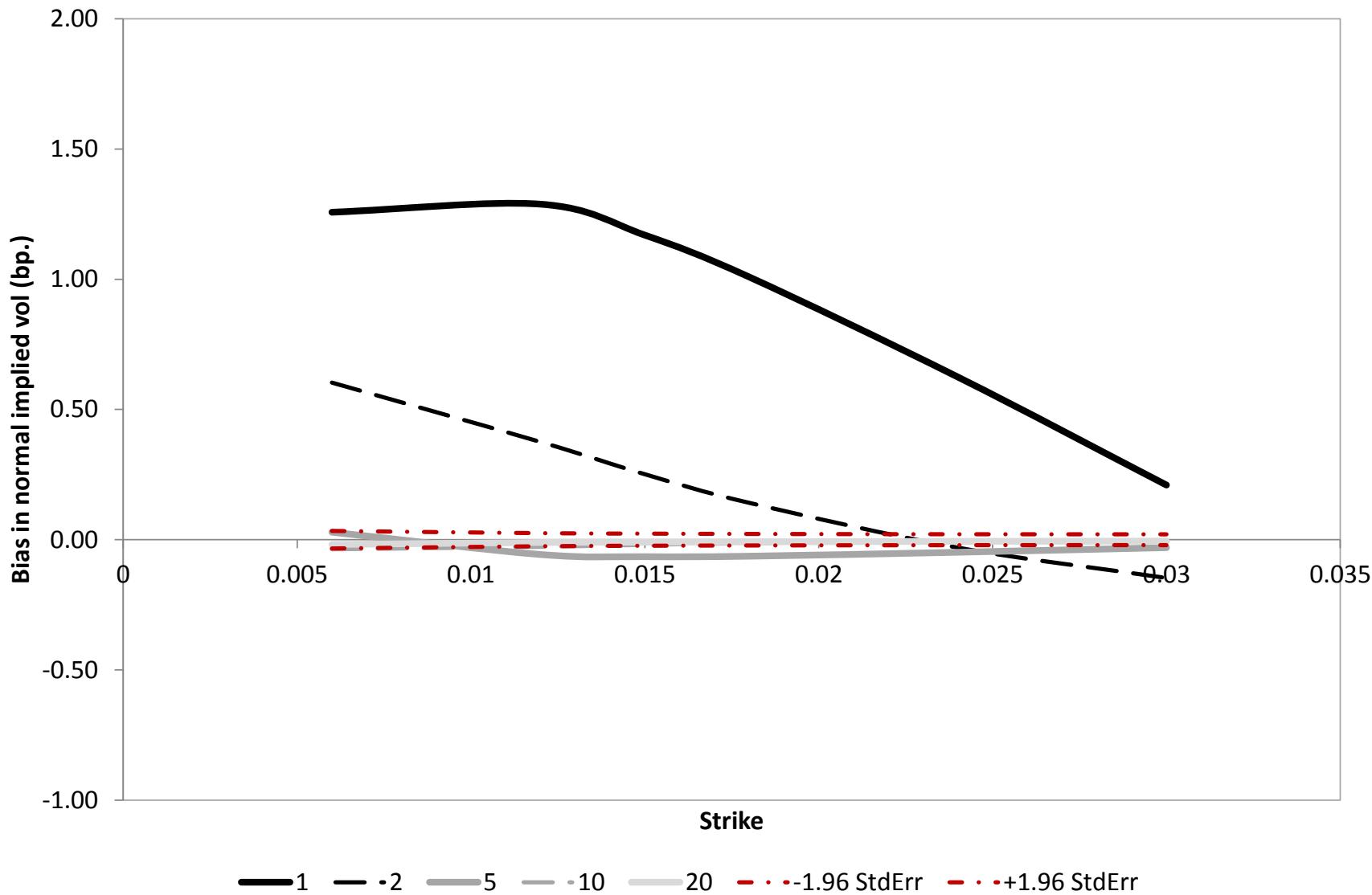
# *Moment matching – NV*



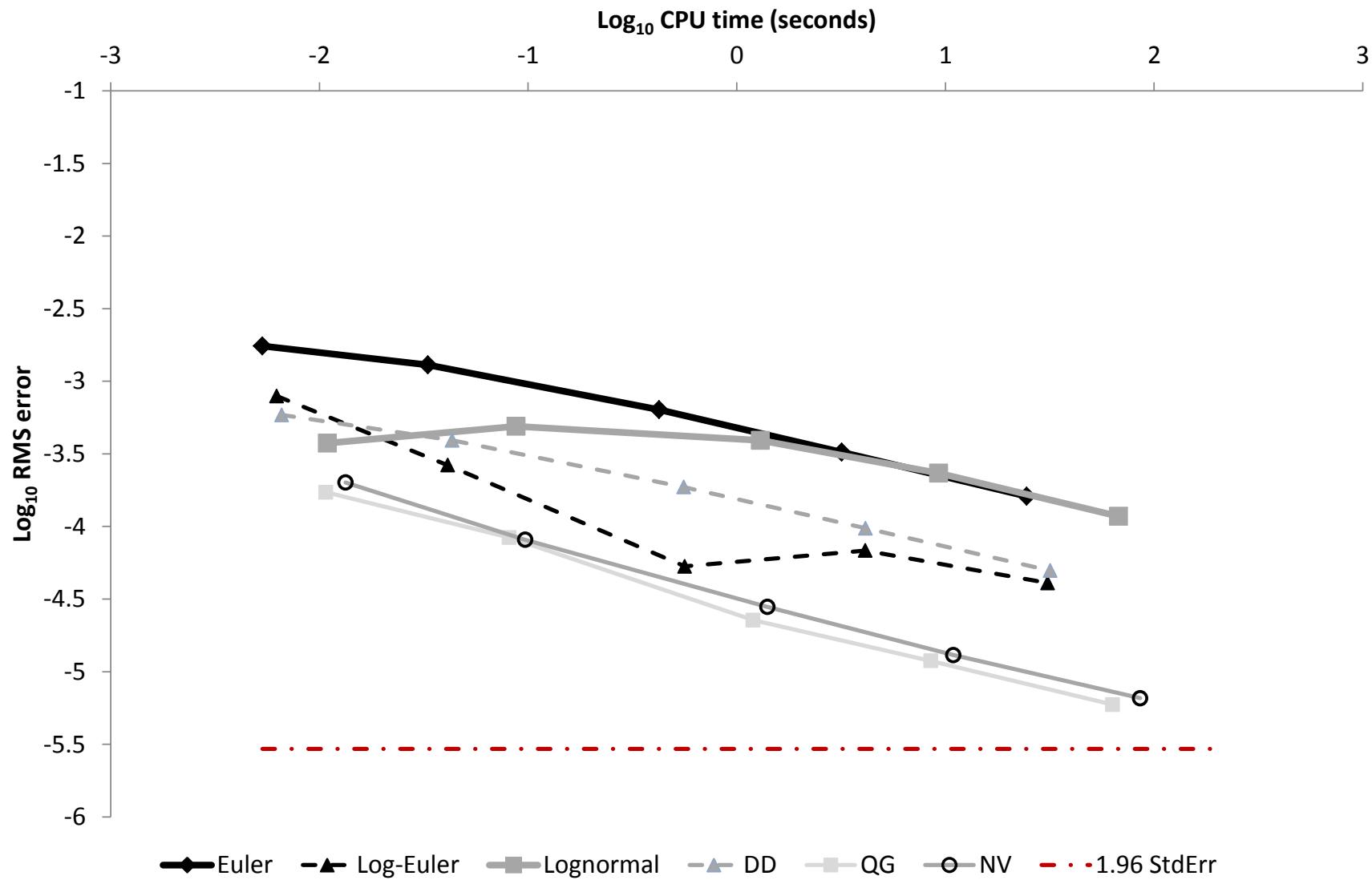
# Moment matching – NV with drift (zoom)



# Moment matching – QG



# Moment matching – RMS error



# *Part IV – Exact simulation of the asset*

# *Exact simulation of the asset*

Instead of using moment matching techniques, we can also use an exact scheme recently devised by Makarov and Glew [2010] for the simulation of squared Bessel processes, with either reflection or absorption.

As a reminder, we are then sampling:

$$dY(t) = 2\sqrt{Y(t)} dW(t) + \delta dt$$

and have to use the time and coordinate transforms to map back to the asset price.

# *Exact simulation of the asset (2)*

We will here use their sequential sampling scheme:

- ▶ With  $p$  as the absorption probability, draw  $U$  from a  $U[0, 1]$  distribution. If  $U < p$ , the process is absorbed.
- ▶ If not absorbed, we sample  $Y(t)$  as:

$$Y(t) \sim \Gamma\left(IG(t) + 1, \frac{1}{2t}\right)$$

where  $IG(t)$  is an incomplete Gamma discrete random variable:

$$IG(t) \sim \Pi\left(\lfloor \frac{\delta-2}{2} \rfloor, \frac{Y(0)}{2t}\right)$$

## *Exact simulation of the asset (3)*

For completeness, a random variable  $X$  follows an  $\text{IG}(\theta, \lambda)$  distribution if it has the following pdf:

$$\mathbb{P}(X = n) = e^{-\lambda} \frac{\lambda^{n+\theta}}{\Gamma(n + \theta + 1)} \frac{\Gamma(\theta)}{\gamma(\theta, \lambda)}$$

Since  $\text{IG}(t)$  is discrete, this means the scale parameter of  $Y(t)$  is integer-valued. We can therefore setup a smart Gamma cache as in Chan and Joshi [2013]. Moreover, we can simplify the distribution as:

$$Y(t) \sim \Gamma\left(\text{IG}(t) + 1, \frac{1}{2t}\right) = \frac{1}{2t} \Gamma\left(\text{IG}(t) + 1, 1\right)$$

## *Exact simulation of the asset (4)*

For large values of  $Y(0) / t$ , which either implies large values of the squared Bessel process, or small values of the time parameter, we alter Makarov and Glew's scheme in order to avoid numerical issues with inverting the incomplete Gamma distribution.

A small time expansion yields:

$$\mathbb{E}[Y(t)] = Y(0) + \delta t + O(t^3)$$

$$\text{Var}(Y(t)) = 2t(\delta t + 2Y(0)) + O(t^3)$$

which we fit to a lognormal distribution.

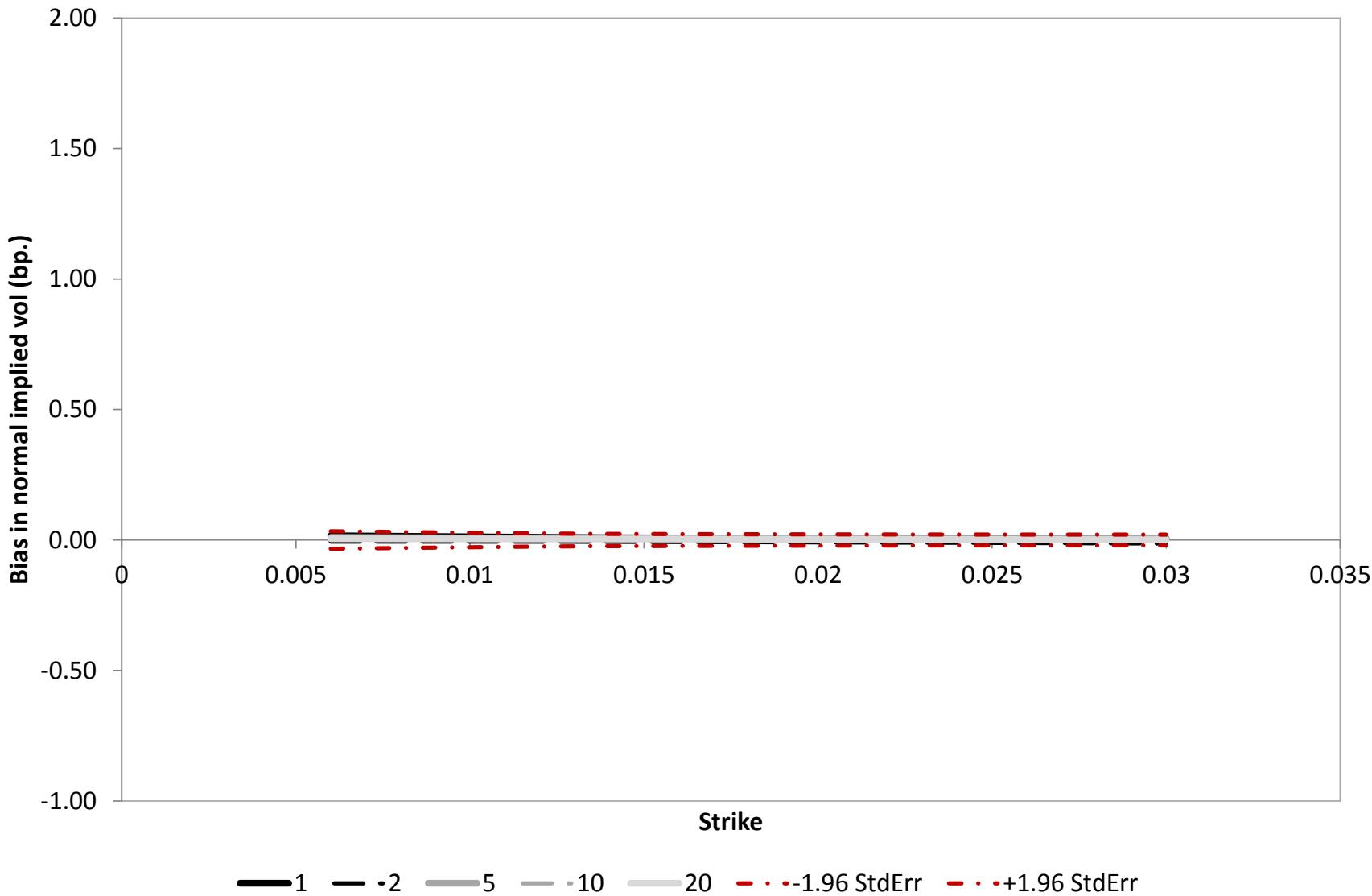
## *Exact simulation of the asset (5)*

If we return to the 10y call in our CEV example:

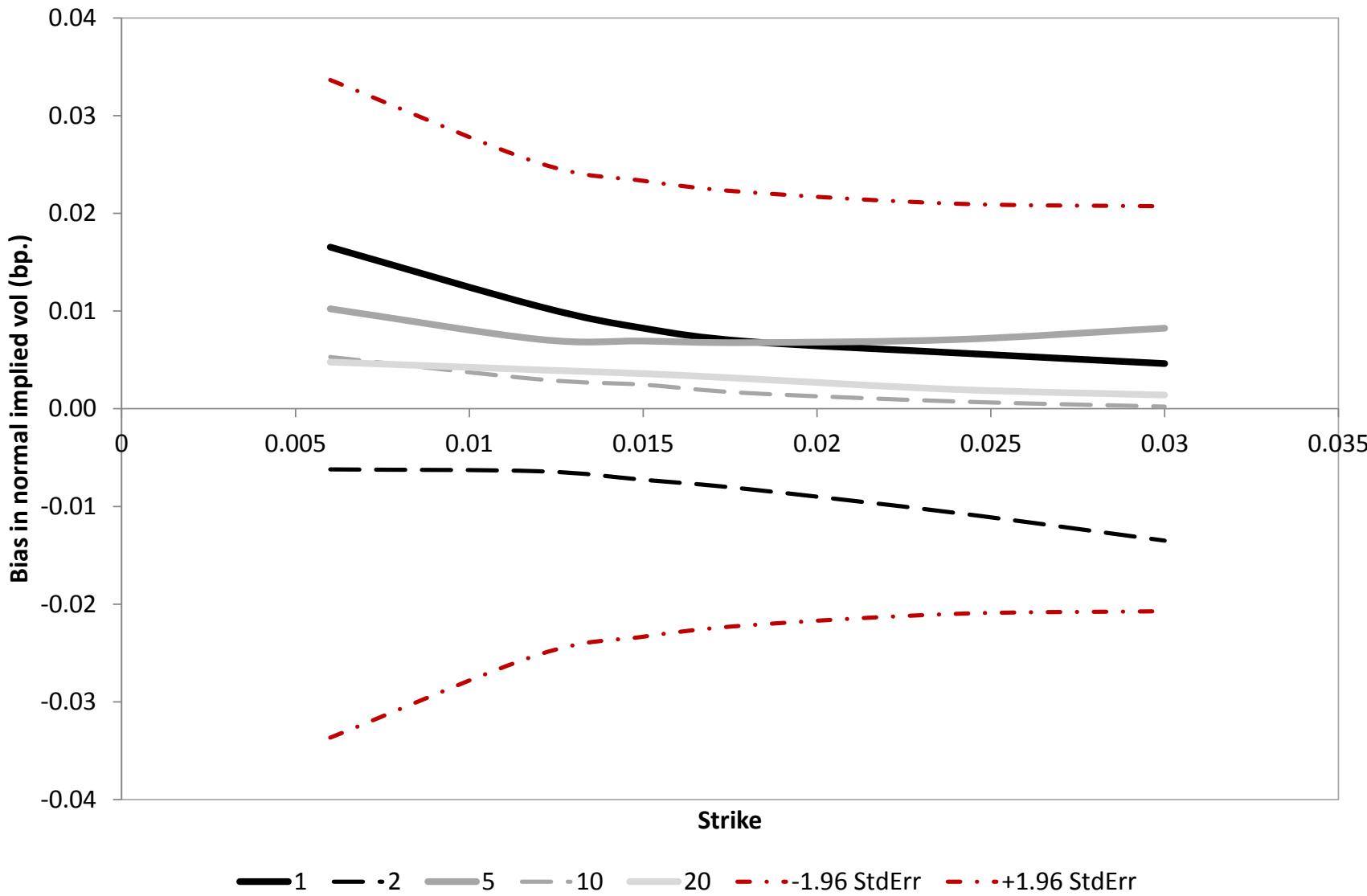
$\sigma$	$\beta$	$S(0)$
5%	0.5	1.5%

we can compare the results to the best-performing moment-matching scheme, the QG scheme.

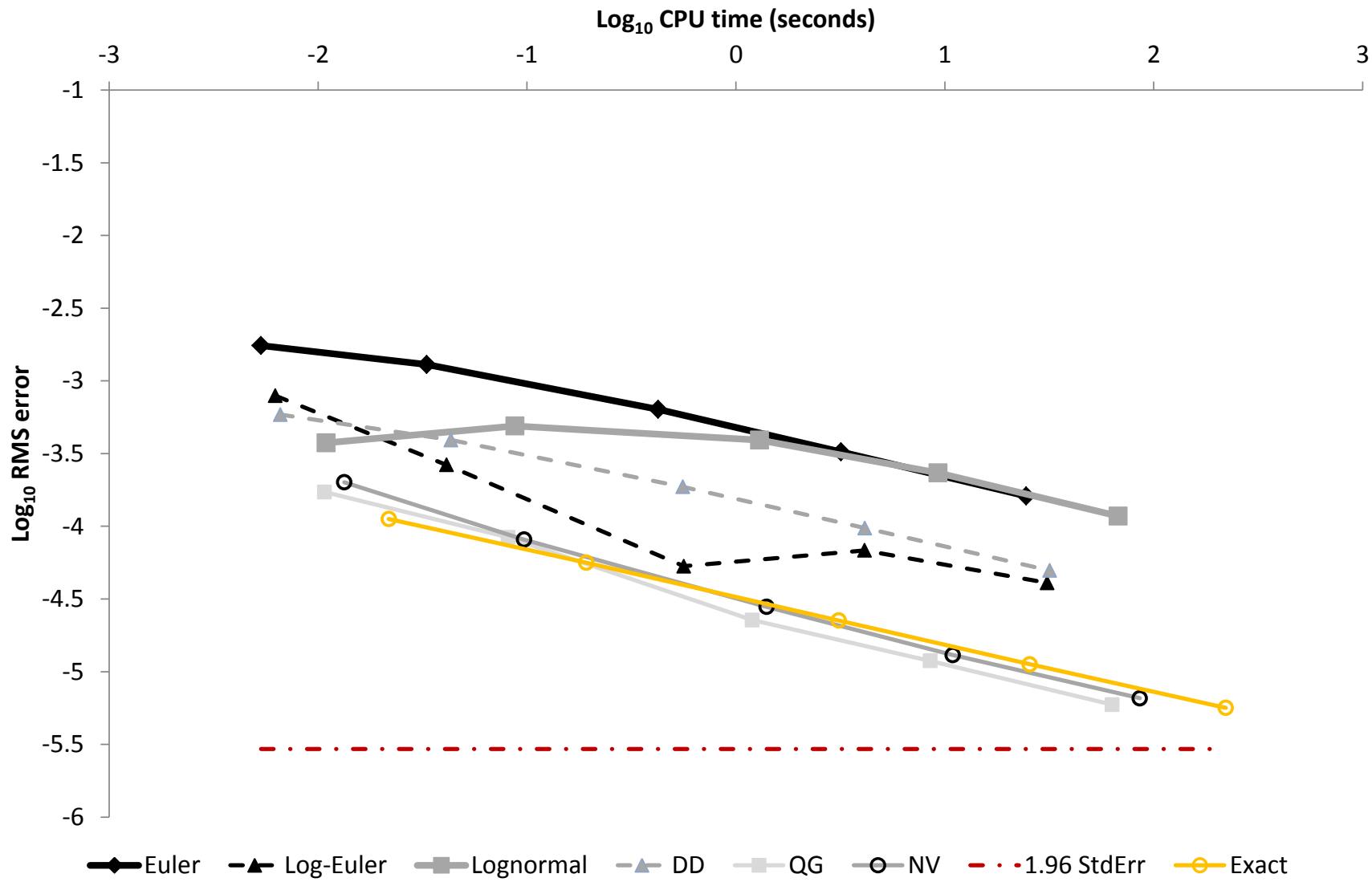
# *Exact simulation of the asset - bias*



# *Exact simulation of the asset – bias (zoom)*



# *Exact simulation of the asset – RMS error*



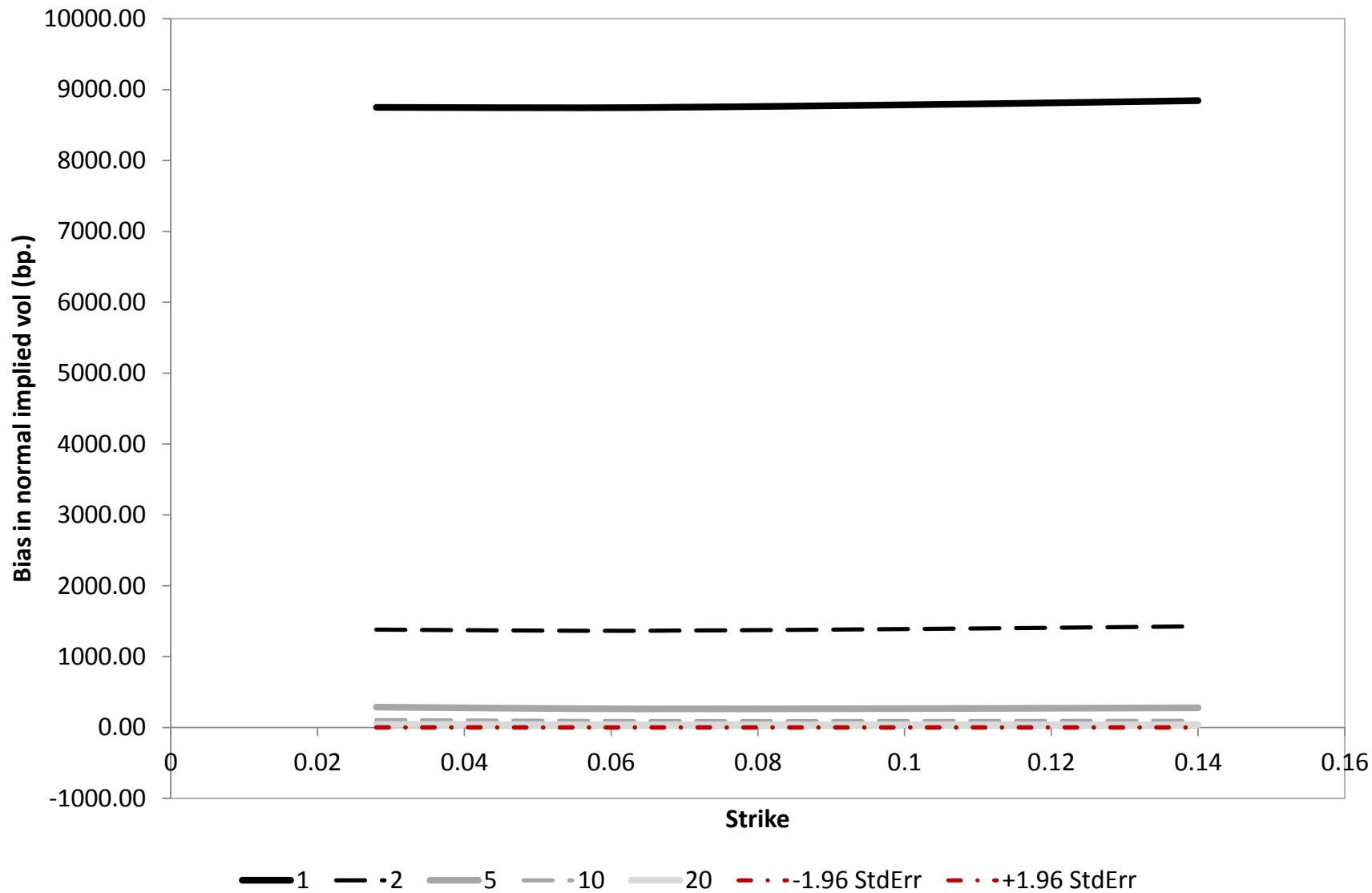
# **SABR $\rho = 0\%$ example**

Now we unleash the best schemes so far on a full-fledged SABR example. It is essentially Case IV of Chen et al. [2012], with the exception that the correlation parameter has been set to zero in order to be able to benchmark against Antonov and Spector's [2012] analytical solution. Case IV was intended to describe volatile market conditions and is therefore rather extreme:

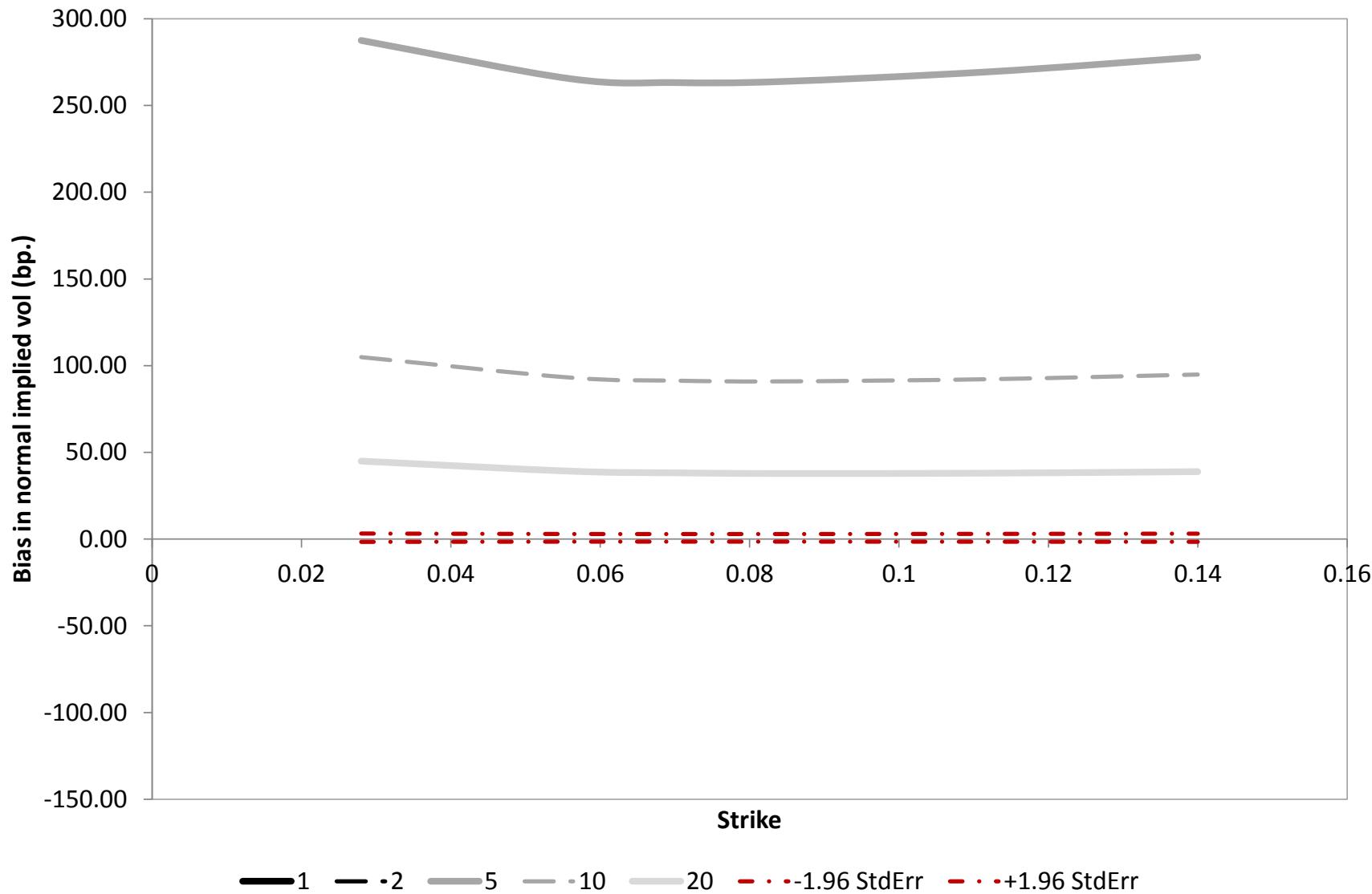
$\sigma(0)$	$\alpha$	$\beta$	$\rho$	$S(0)$
40%	80%	0.5	0%	7%

We focus on a 5y European call option.

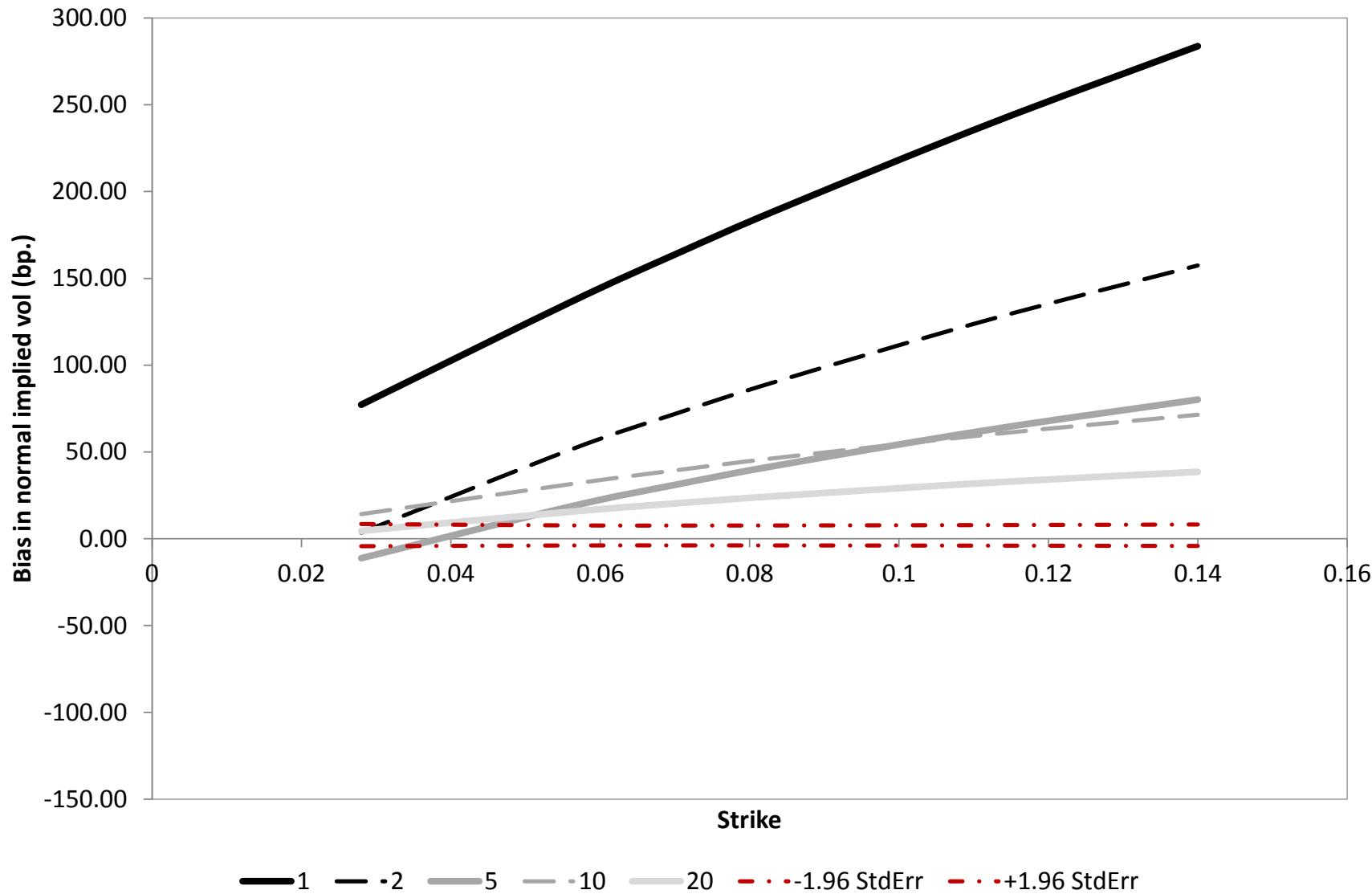
# *SABR $\rho = 0\%$ – NV with drift*



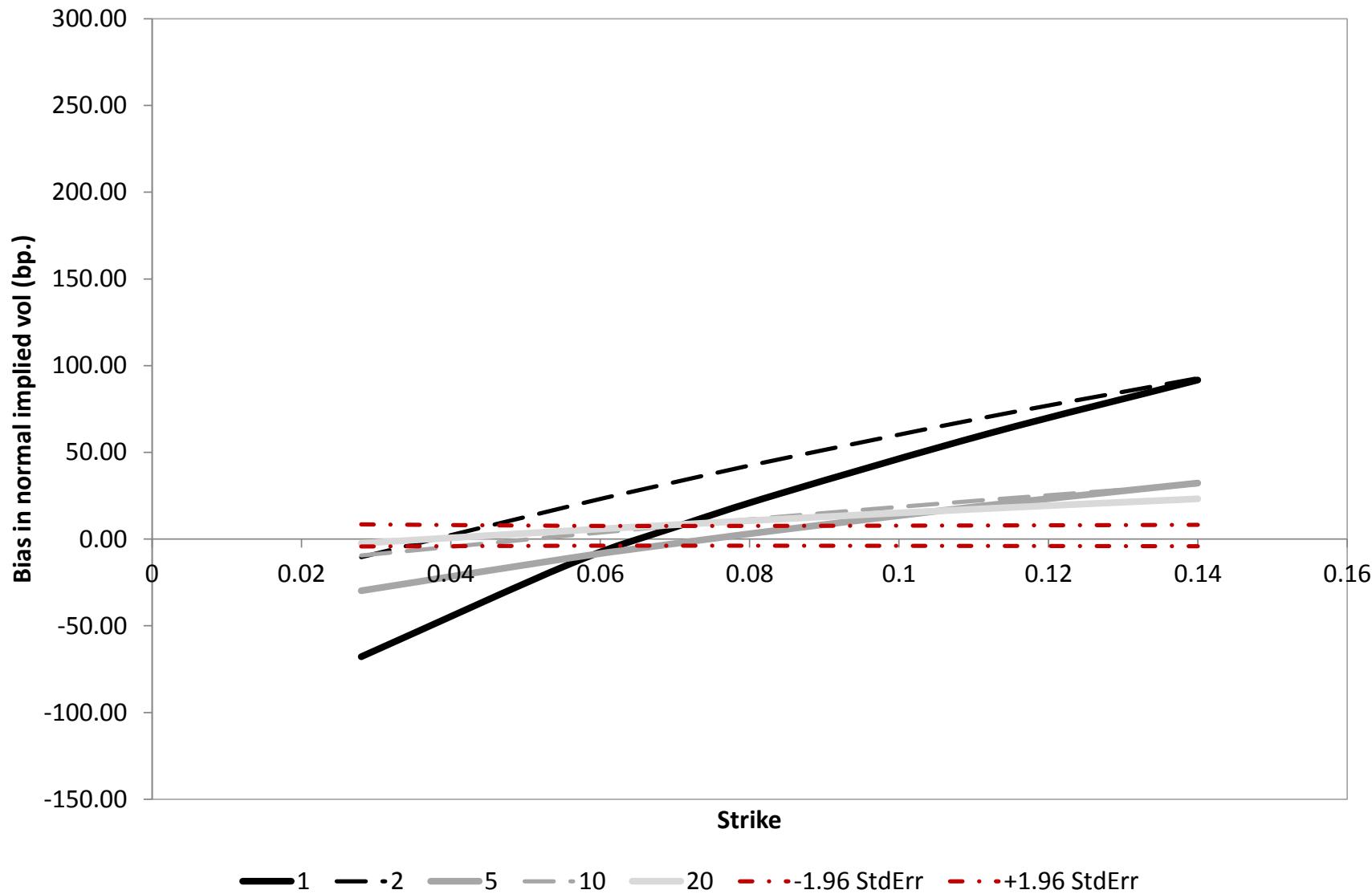
# *SABR $\rho = 0\%$ – NV with drift (zoom)*



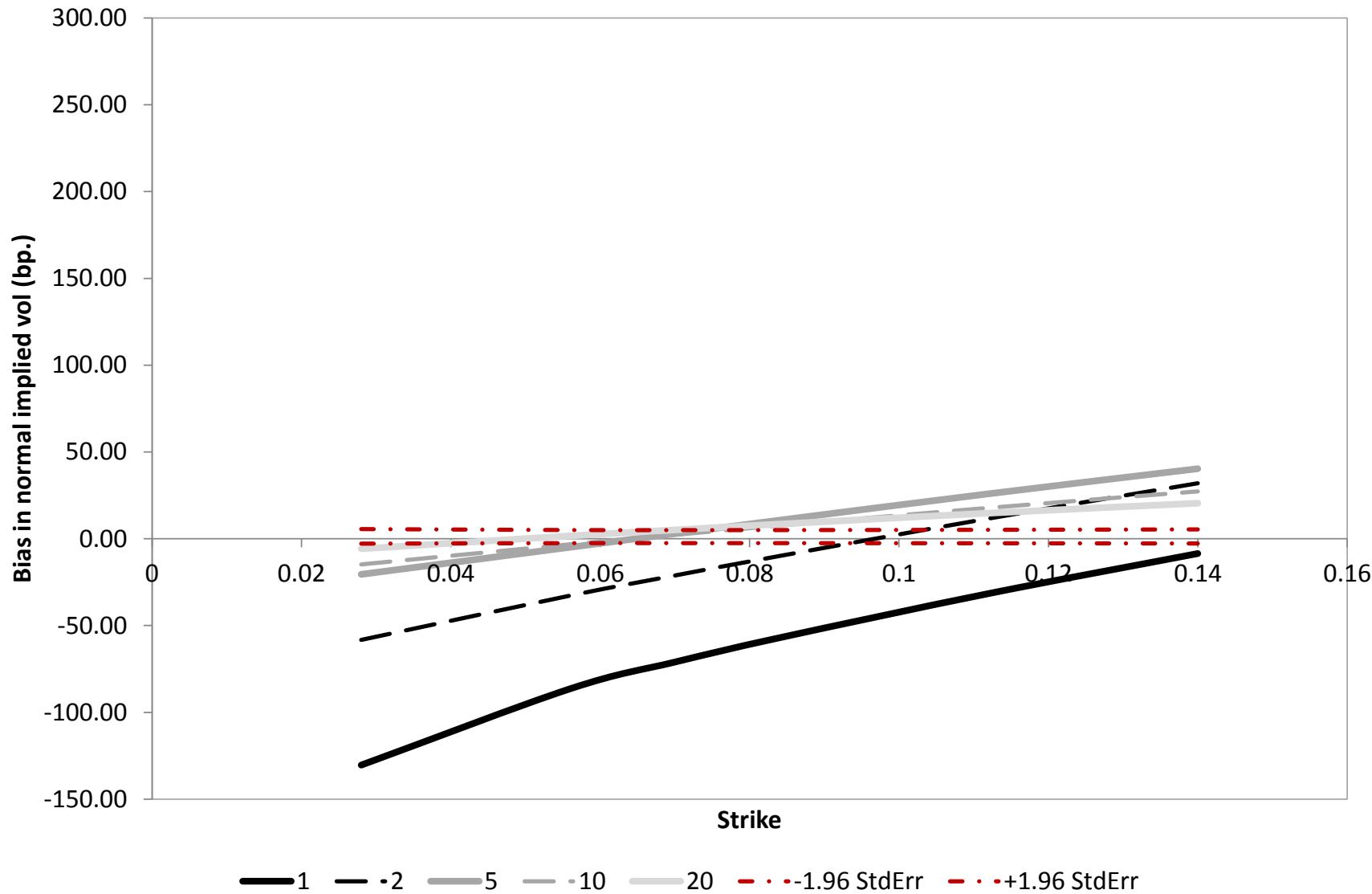
# *SABR $\rho = 0\%$ – Log-Euler*



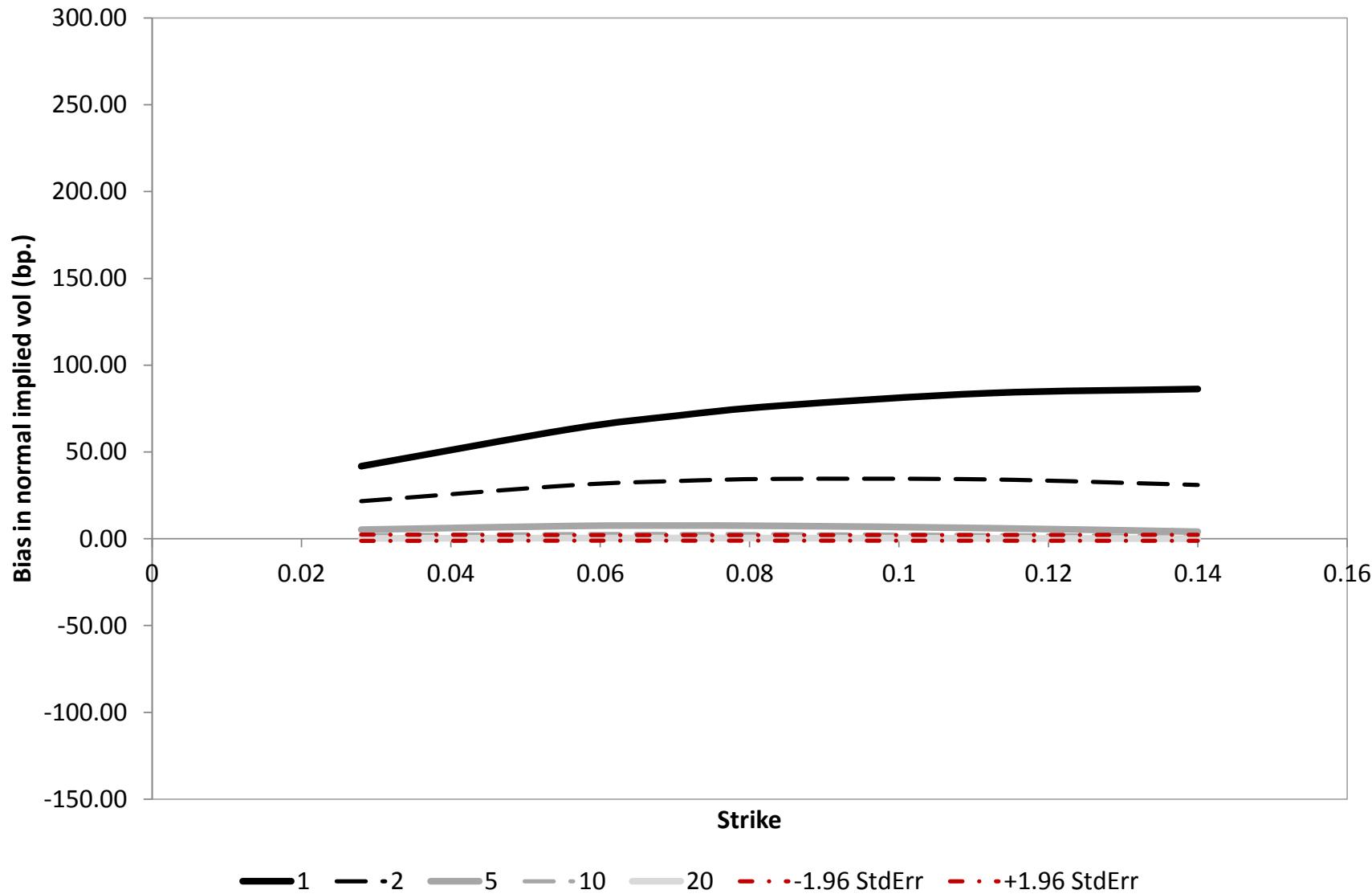
# *SABR $\rho = 0\%$ – Log-Euler+/Central*



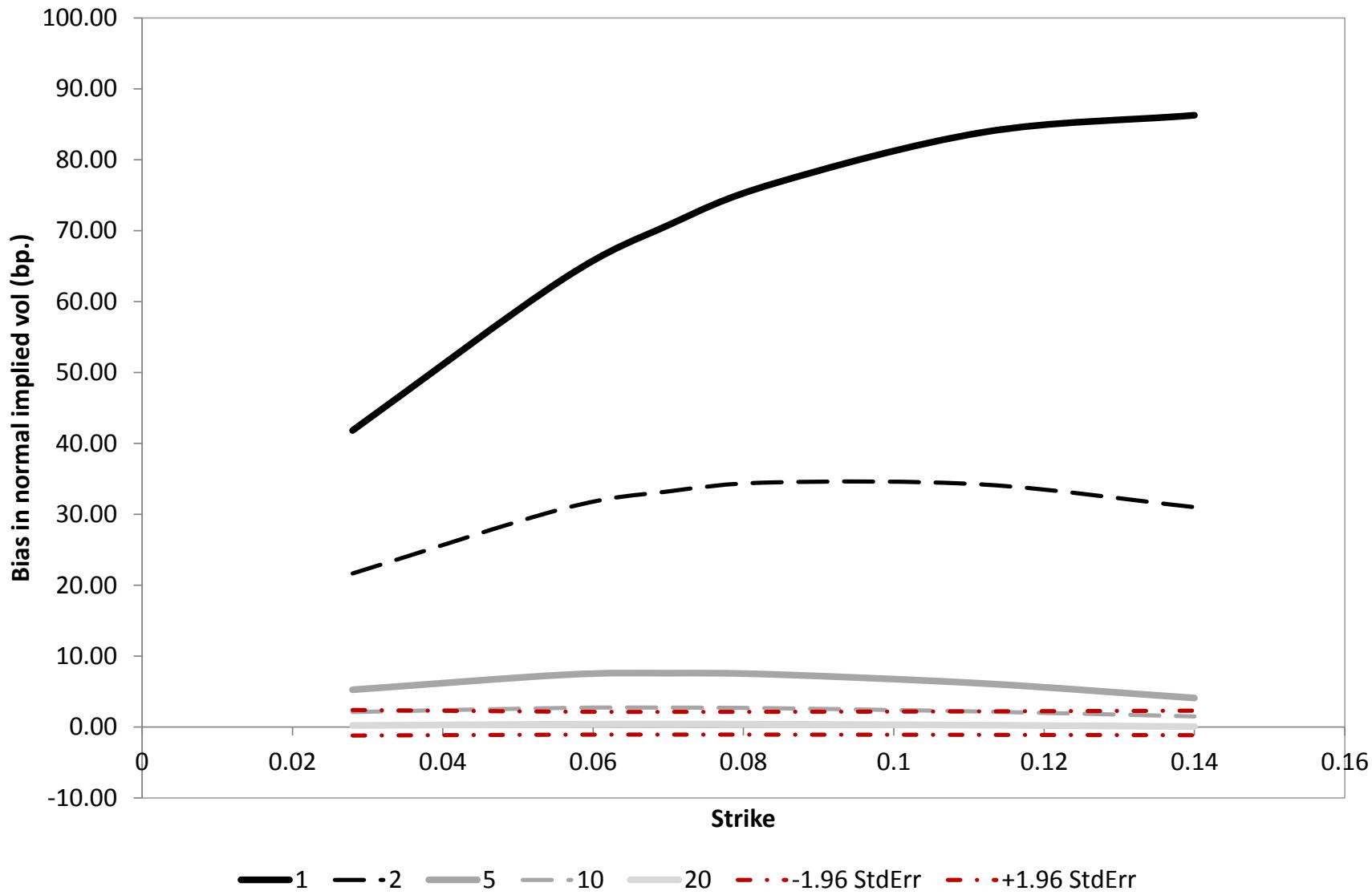
# *SABR $\rho = 0\%$ – Log-Euler+/Curran 1M+*



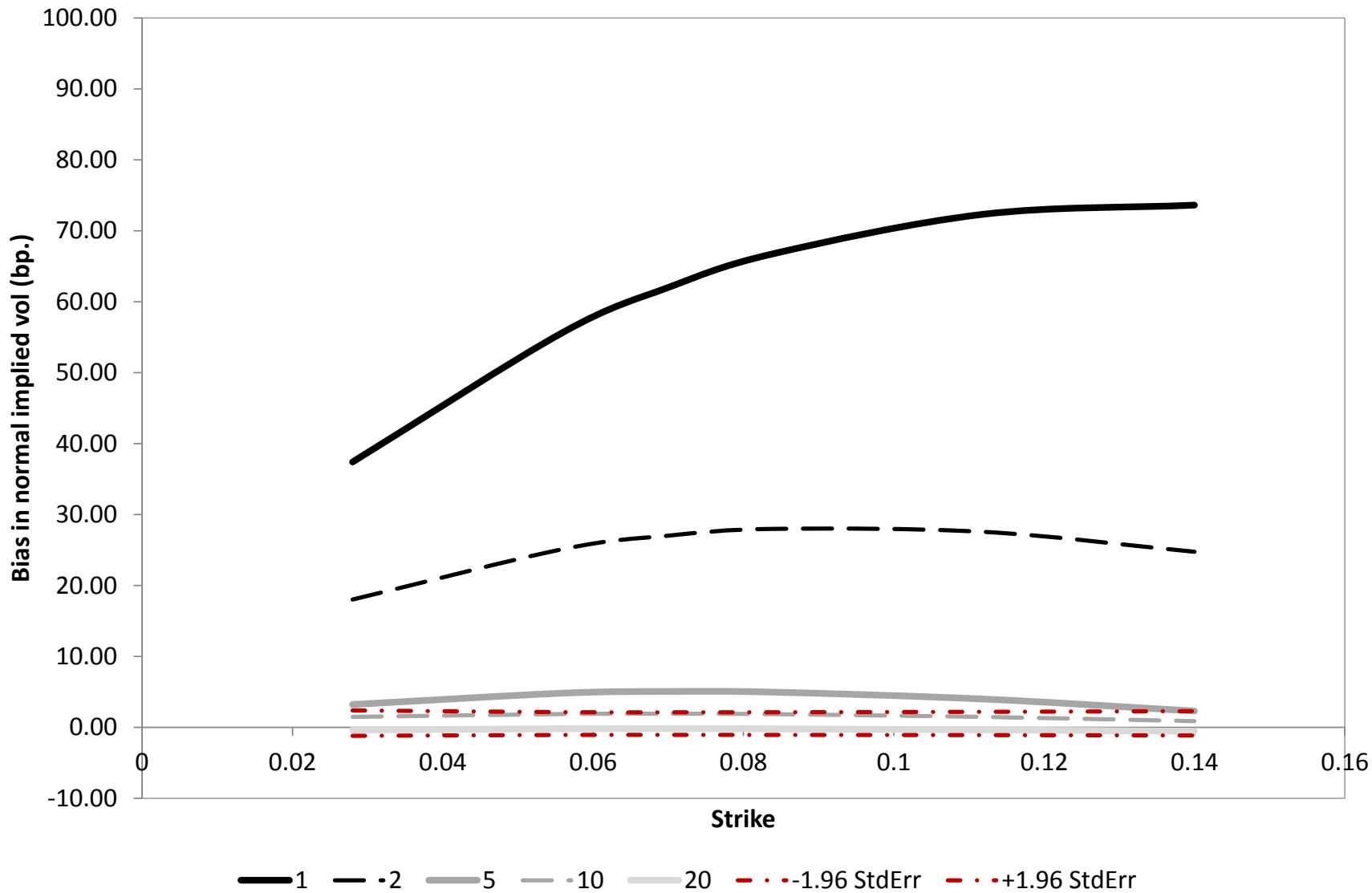
# *SABR $\rho = 0\%$ – QG/Central*



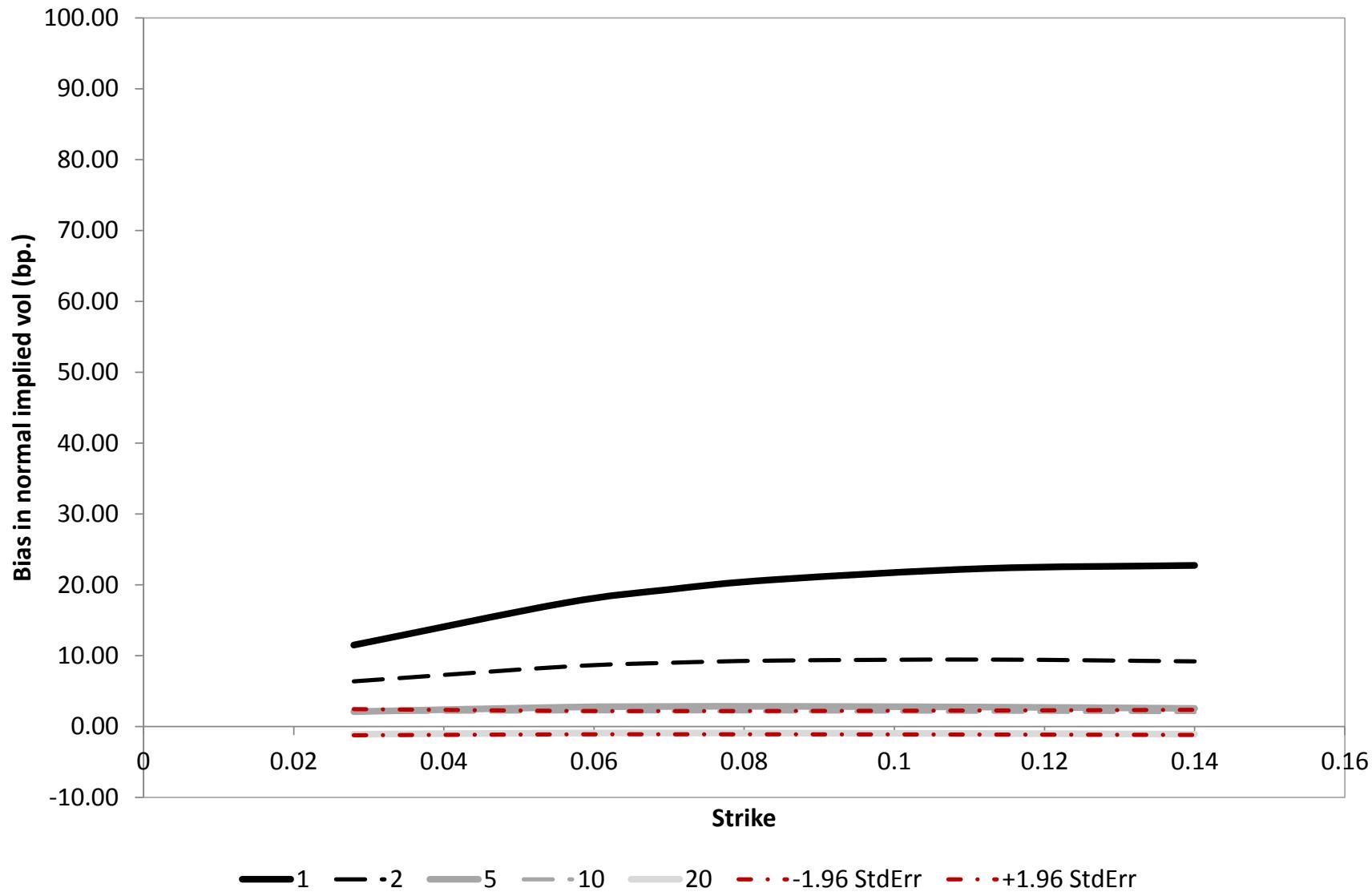
# *SABR $\rho = 0\%$ – QG/Central (zoom)*



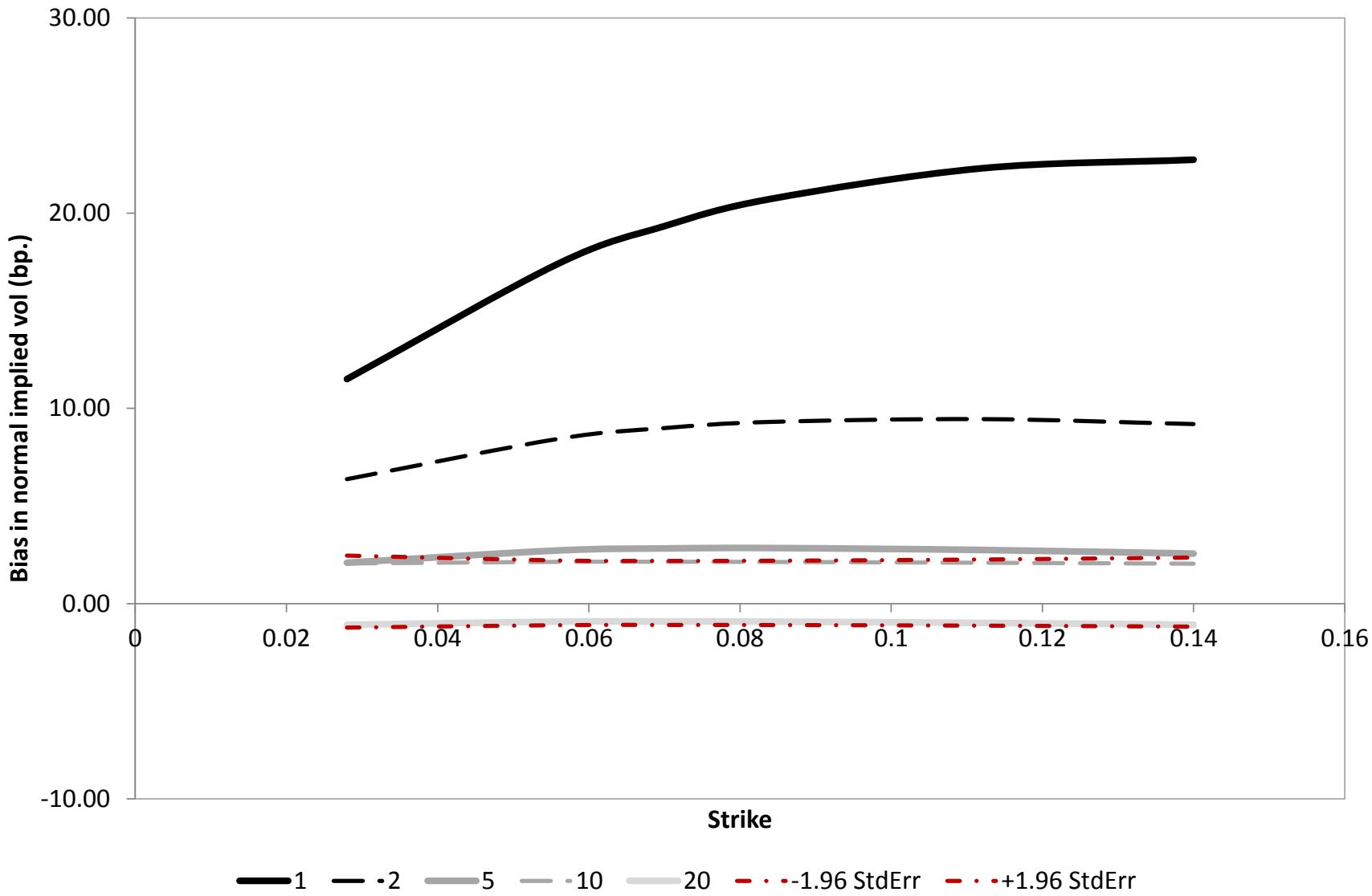
# *SABR $\rho = 0\%$ – Exact/Central*



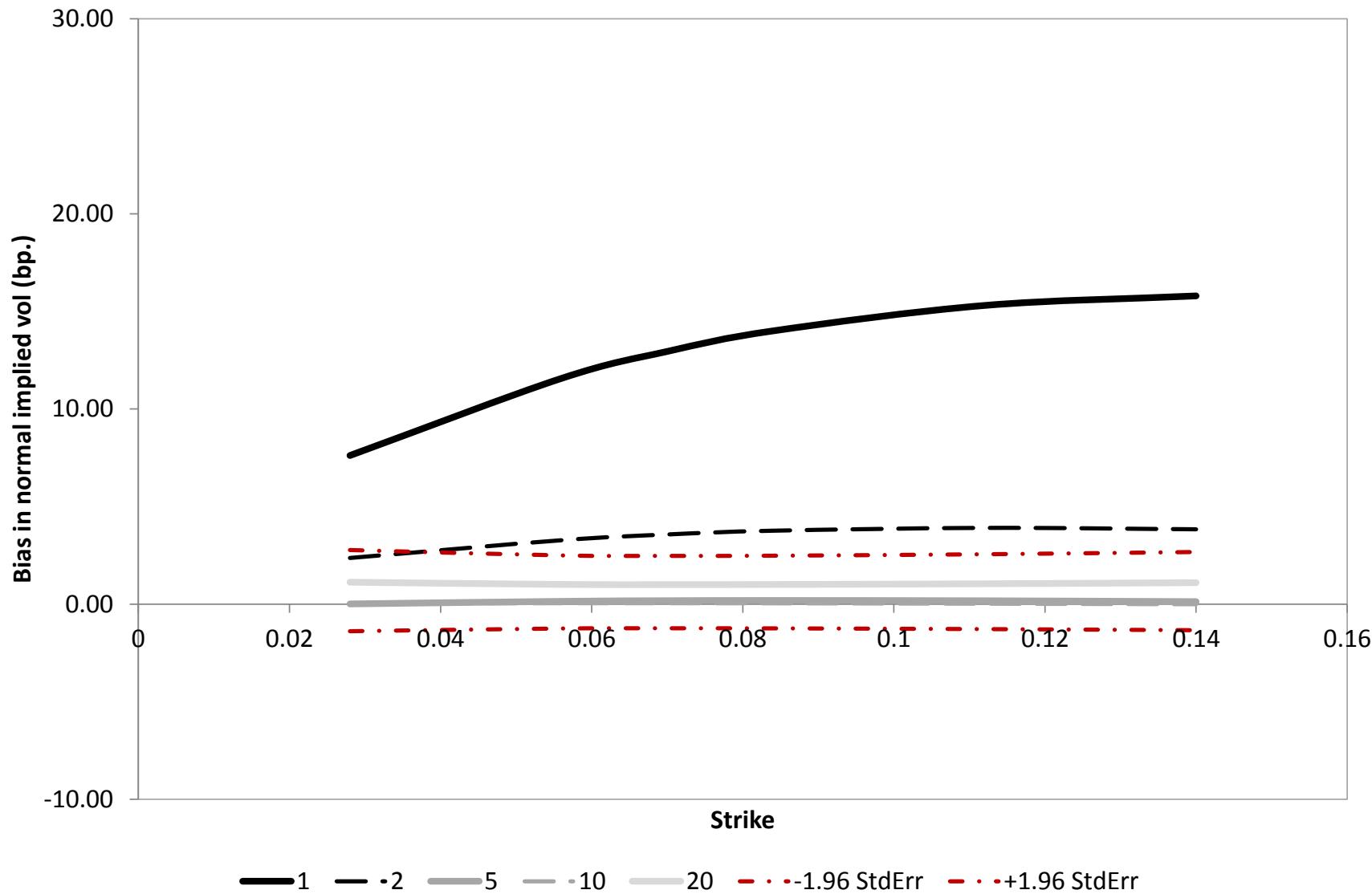
# *SABR $\rho = 0\%$ – QG/Curran 1M+*



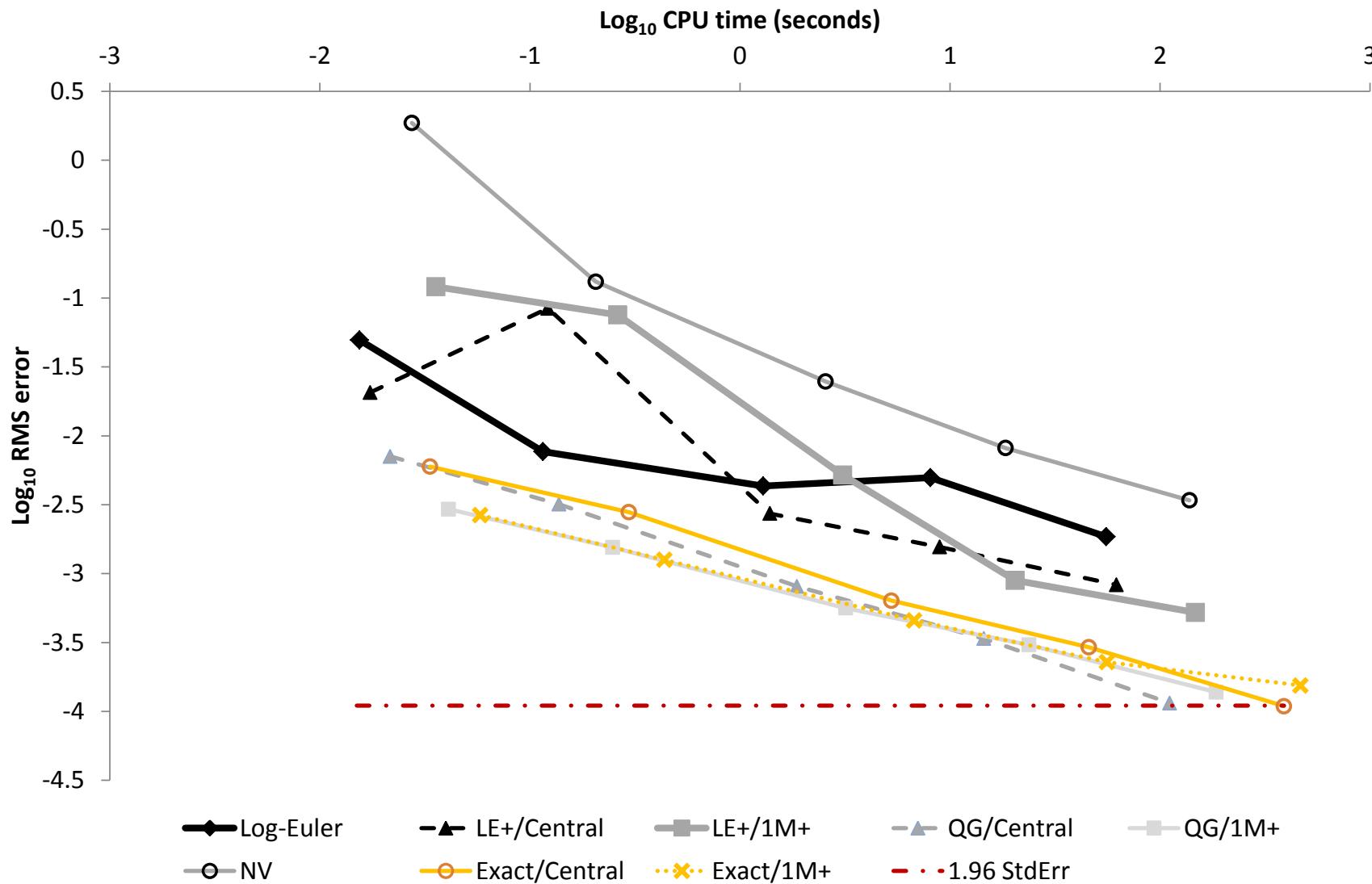
# *SABR $\rho = 0\%$ – QG/Curran 1M+ (zoom)*



# *SABR $\rho = 0\%$ – Exact/Curran 1M+*



# $SABR \rho = 0\% - RMS \text{ error}$



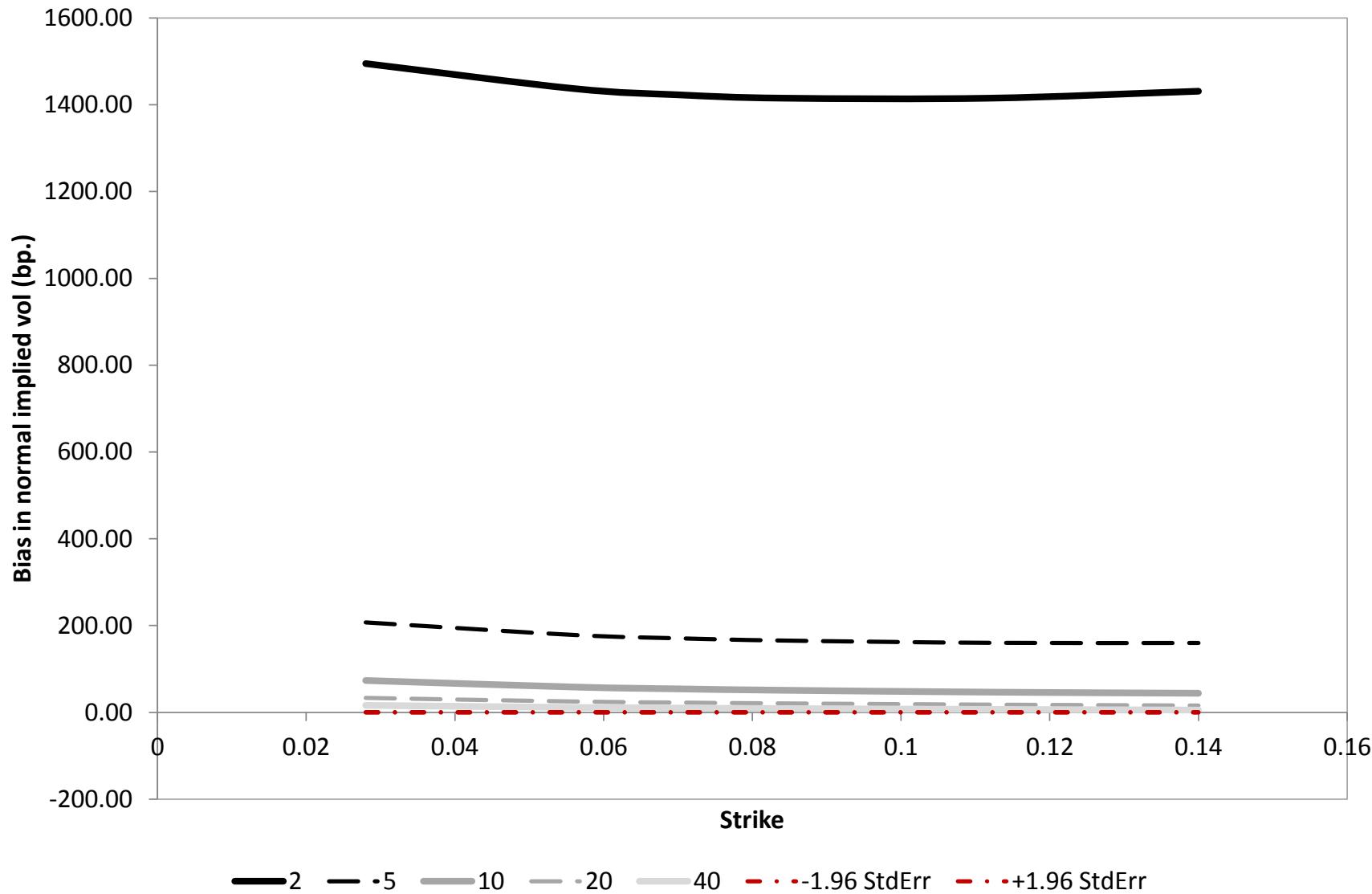
# *SABR $\rho = -60\%$ example*

To gauge the quality of Islah's approximation in case of non-zero correlation, we now consider the original Case IV of Chen et al. [2012]:

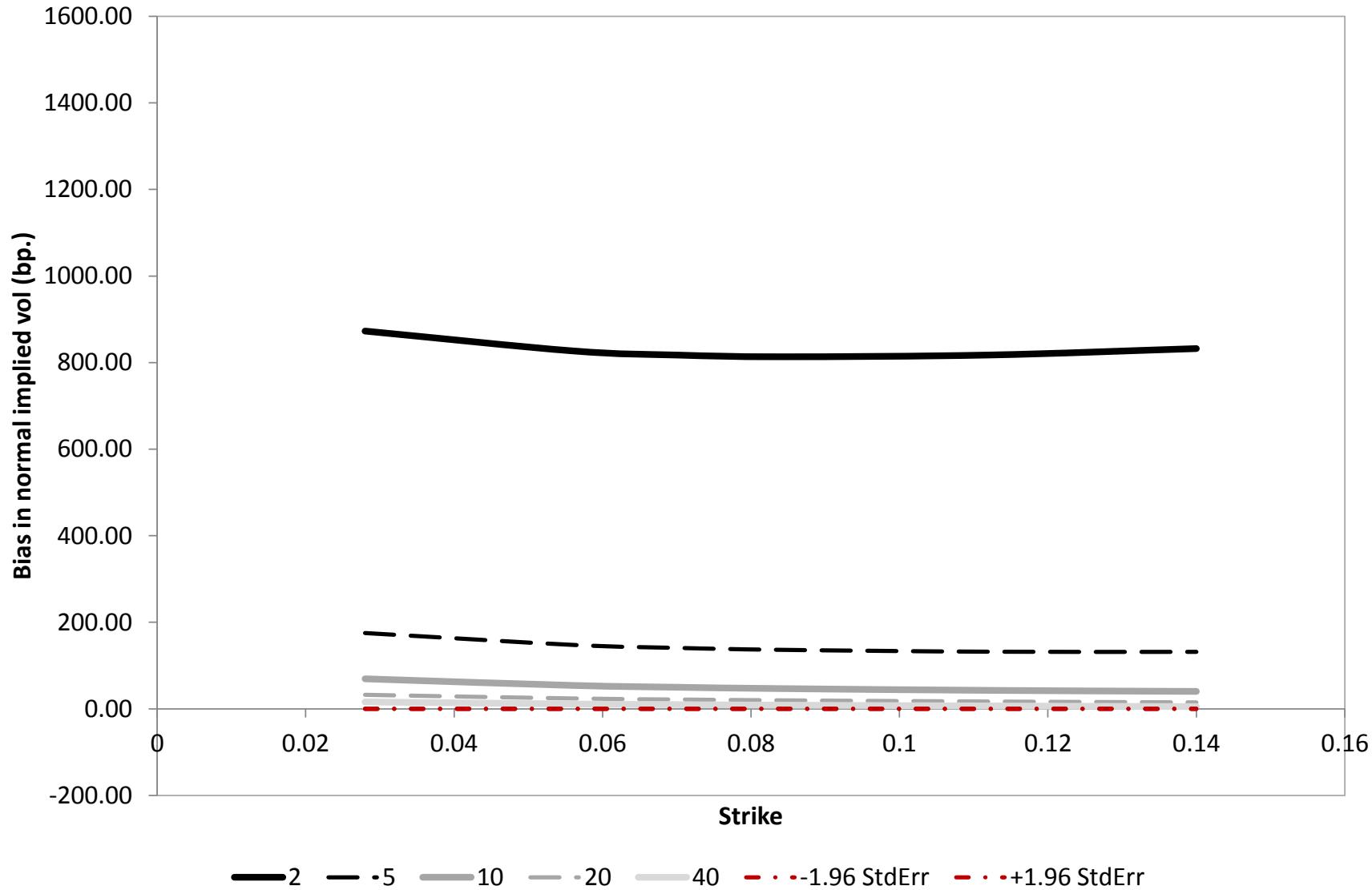
$\sigma(0)$	$\alpha$	$\beta$	$\rho$	$S(0)$
40%	80%	0.5	-60%	7%

and again focus on a 5y European call option.

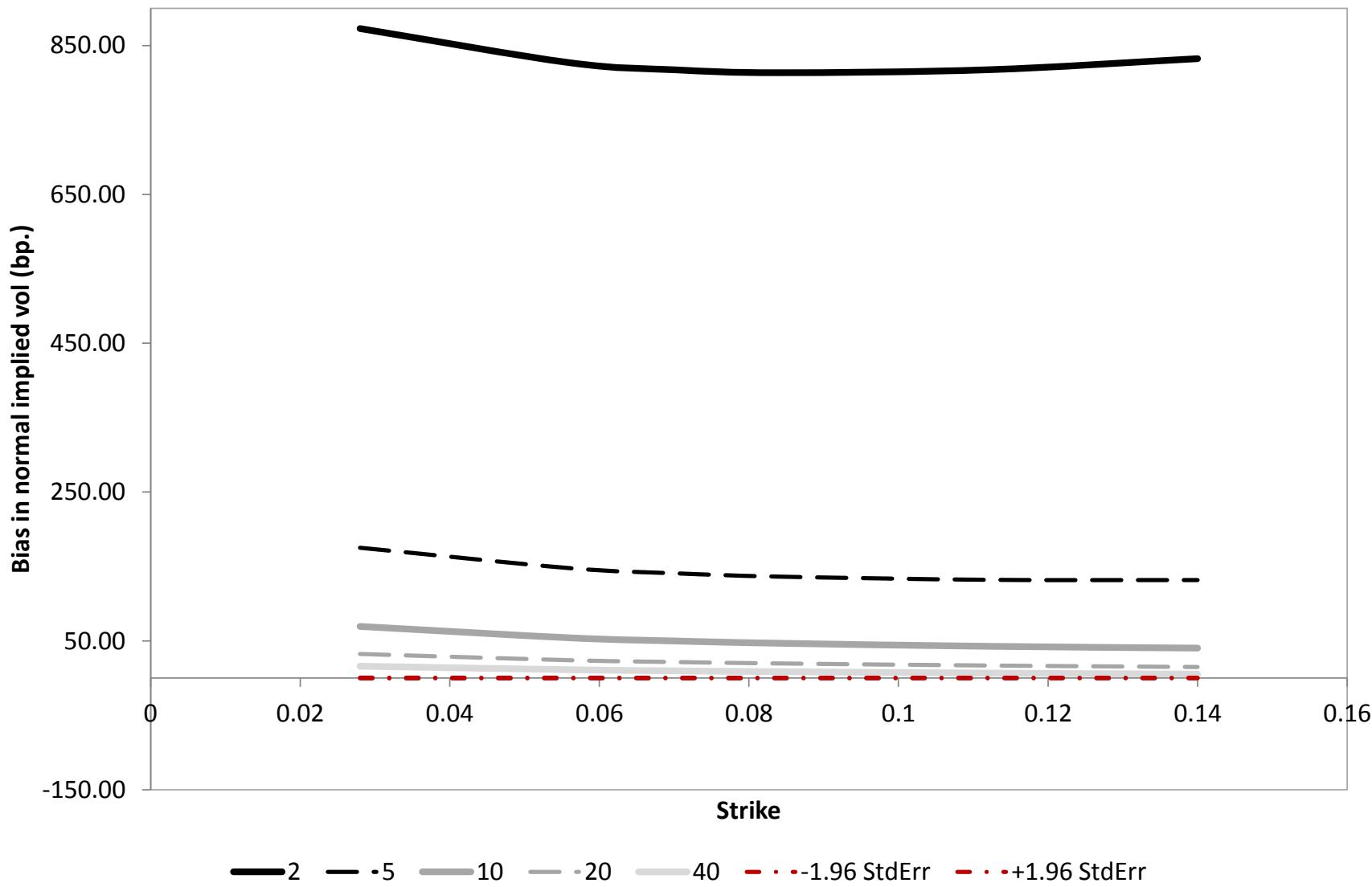
# *SABR $\rho = -60\%$ – ‘Exact’/1M+*



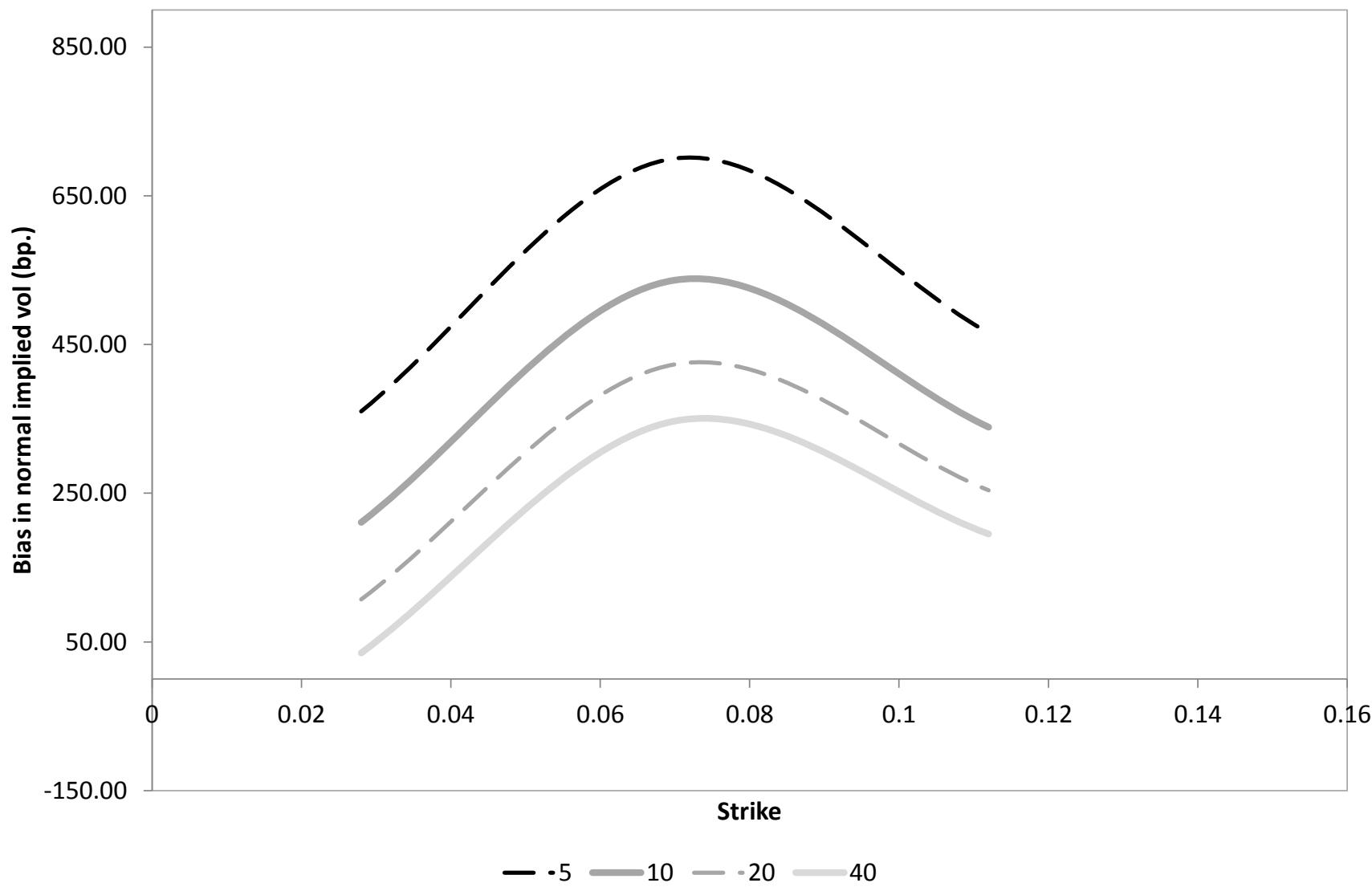
# *SABR $\rho = -60\%$ – ‘Exact’/Central*



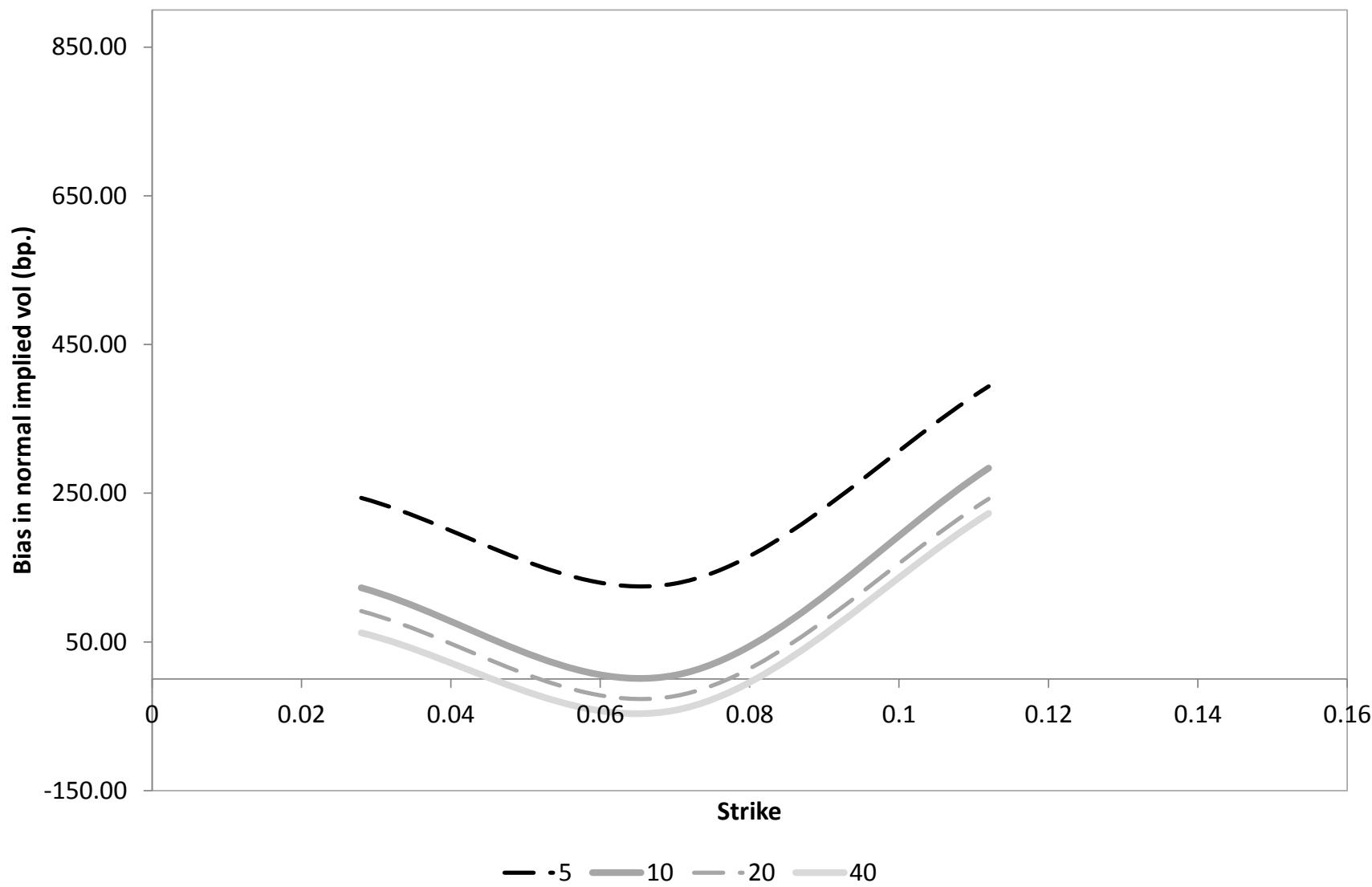
# *SABR $\rho = -60\%$ – ‘Exact’/Central (zoom)*



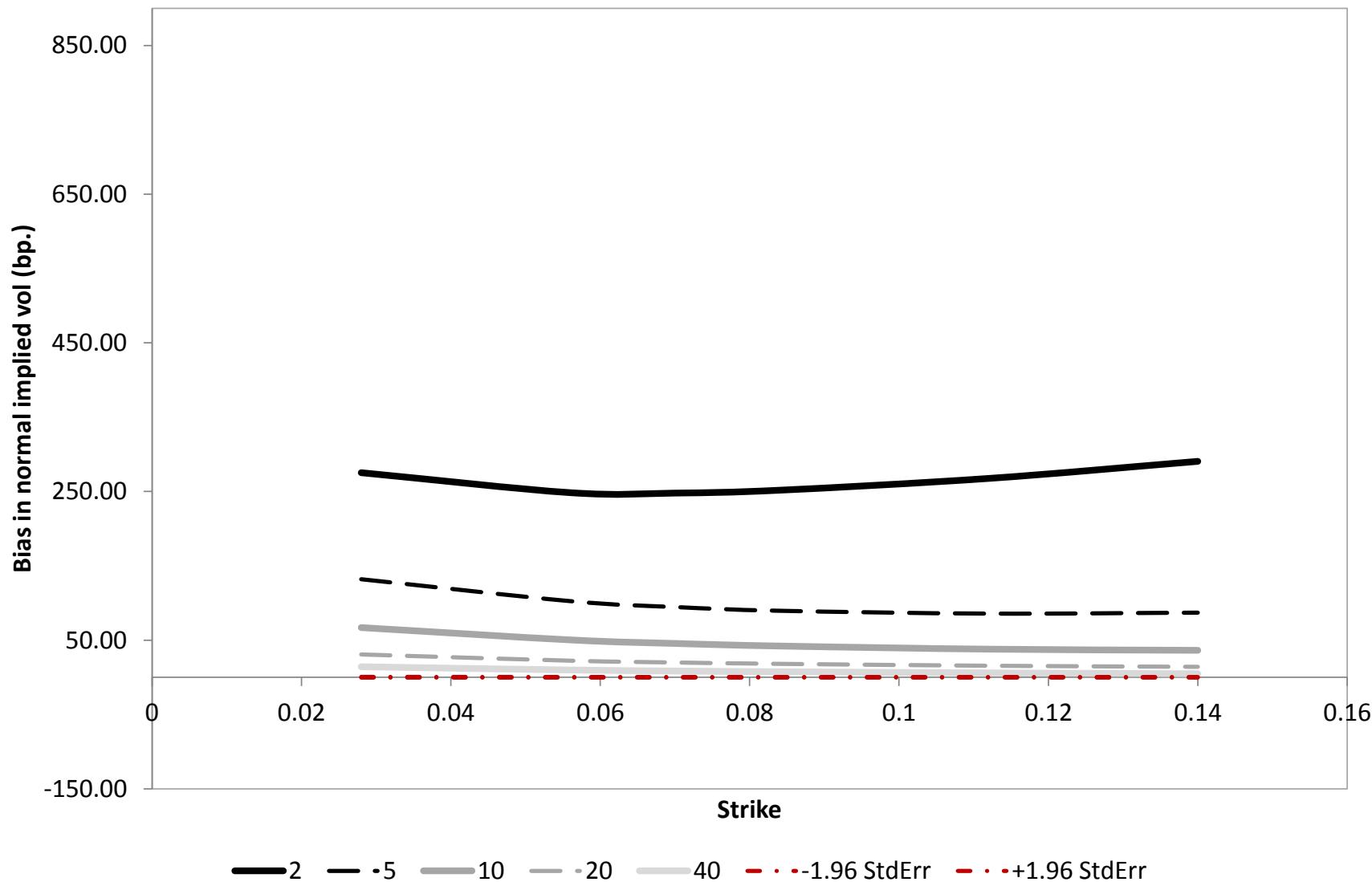
# *SABR $\rho = -60\%$ – Chen et al.'s Euler*



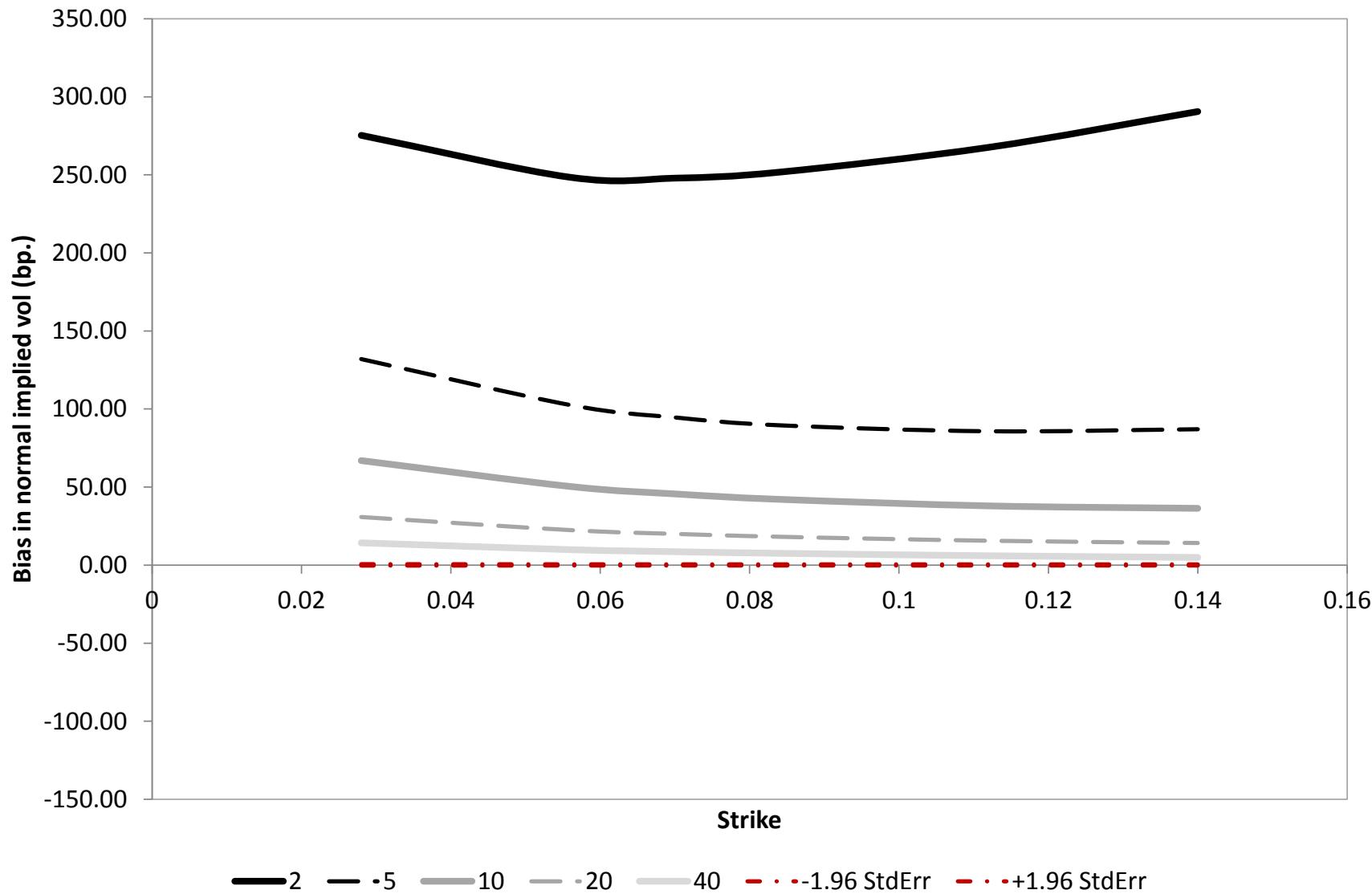
# *SABR $\rho = -60\%$ – Chen et al.'s low-bias*



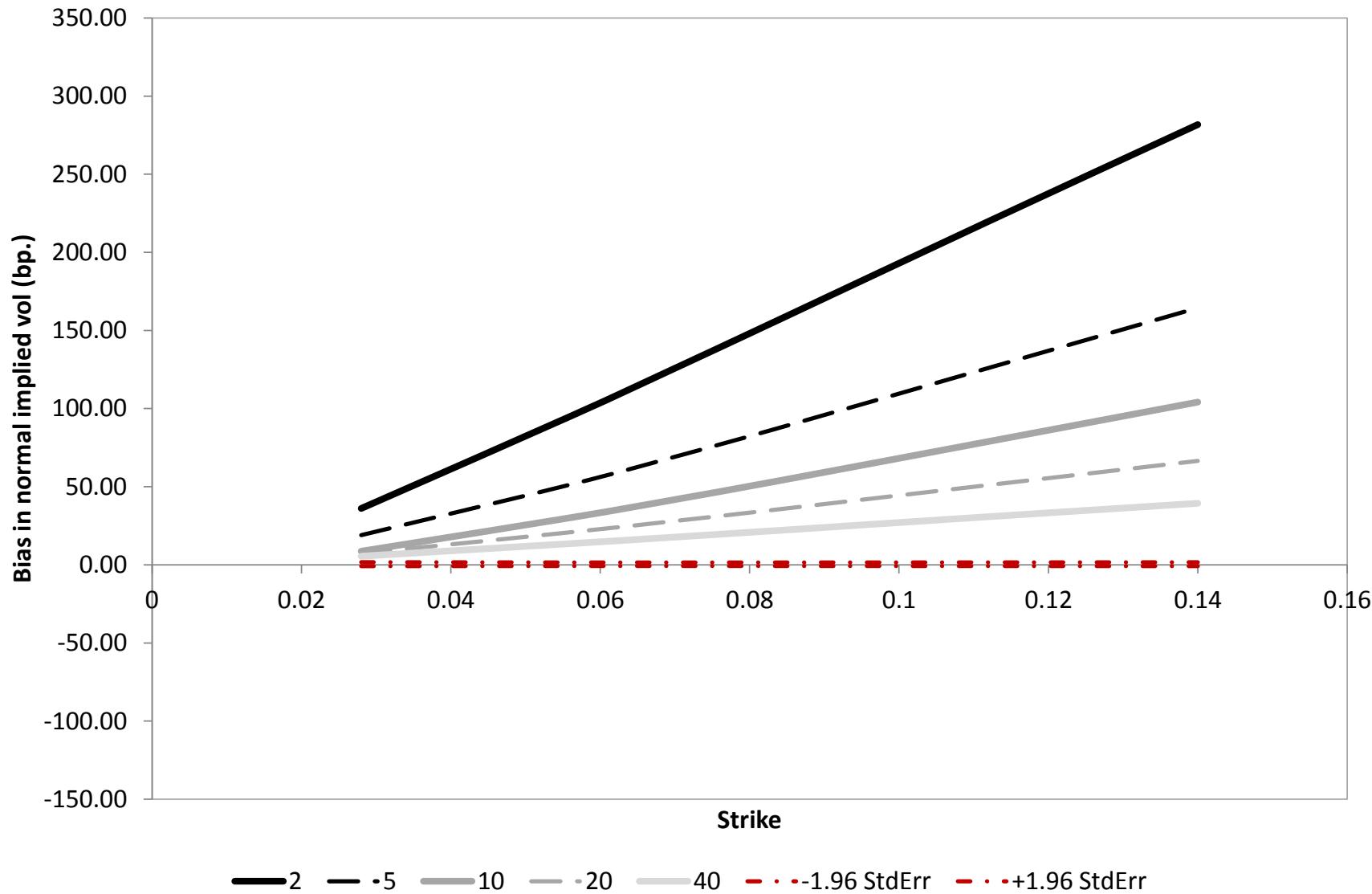
# *SABR $\rho = -60\%$ – NV with drift*



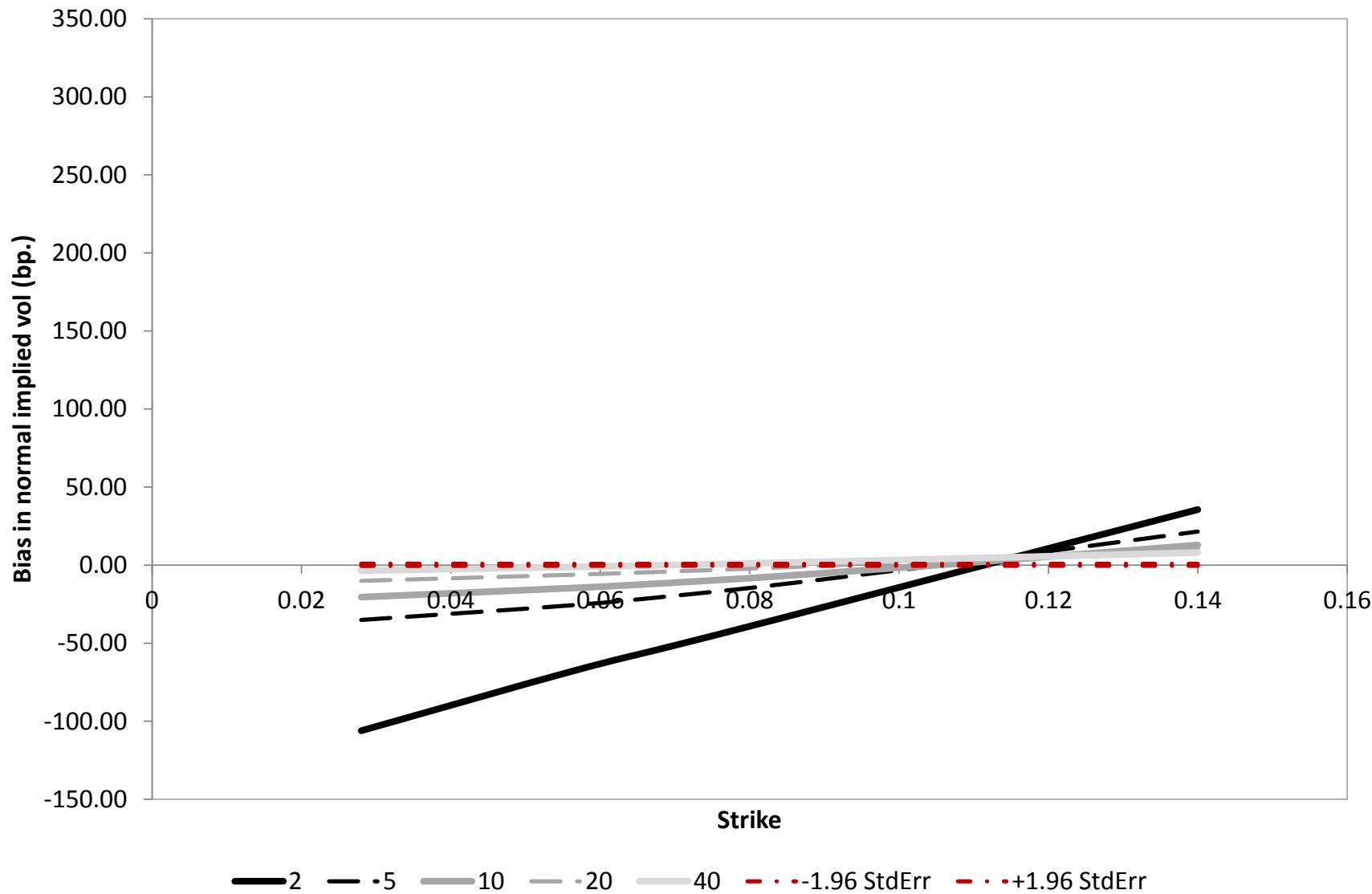
# *SABR $\rho = -60\%$ – NV with drift (zoom)*



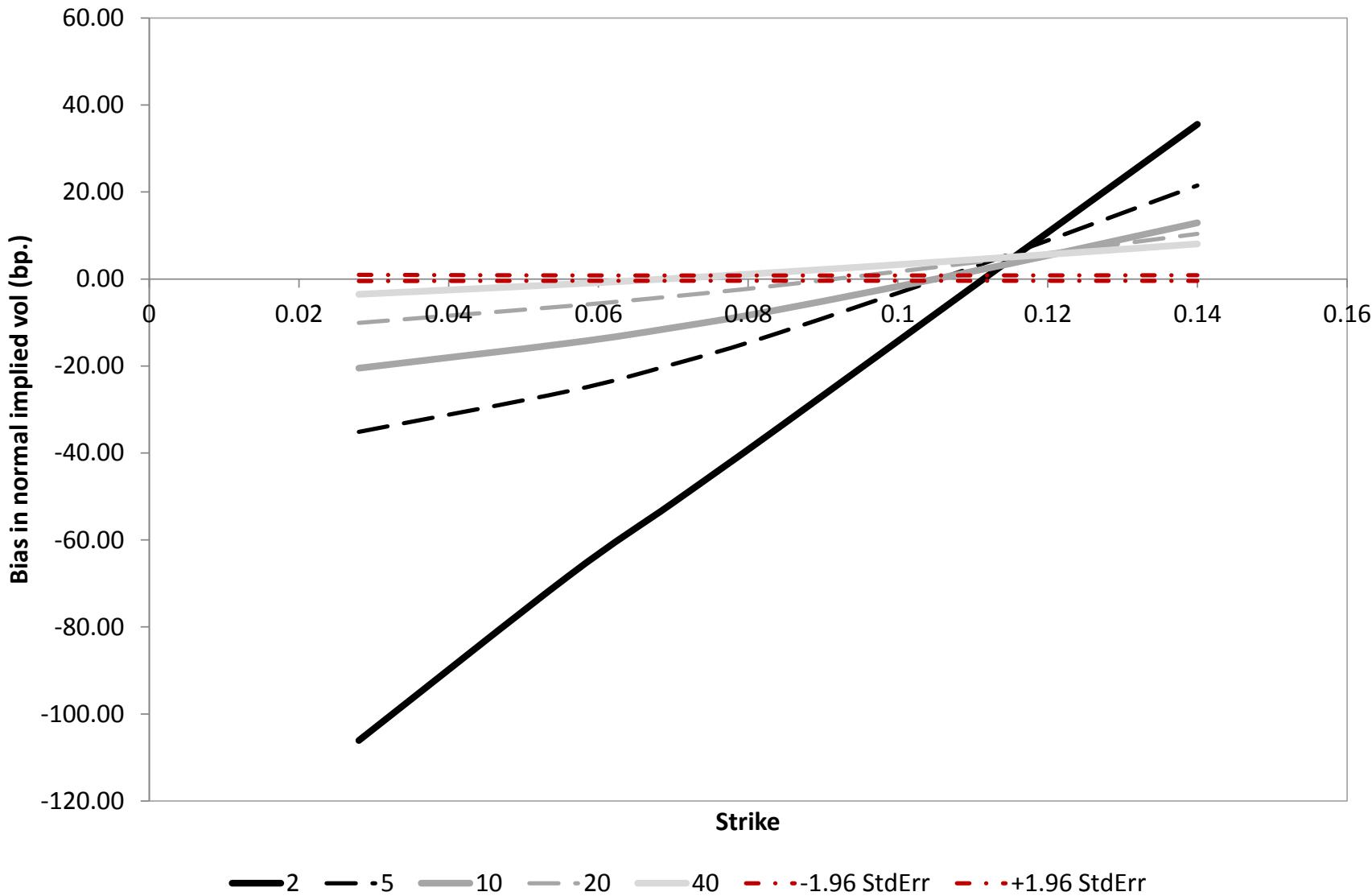
# *SABR $\rho = -60\%$ – Log-Euler*



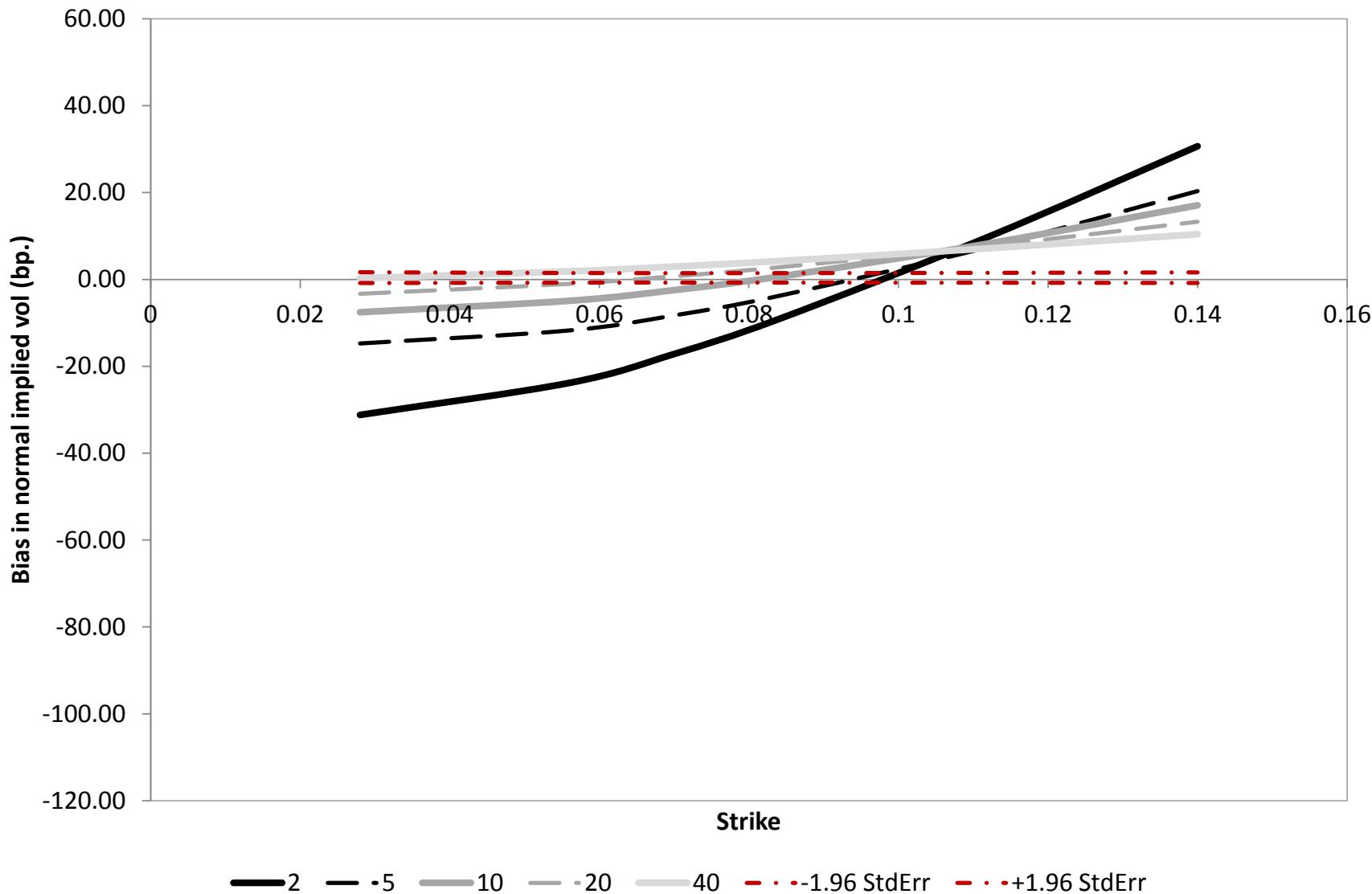
# *SABR $\rho = -60\%$ – Log-Euler+/Central*



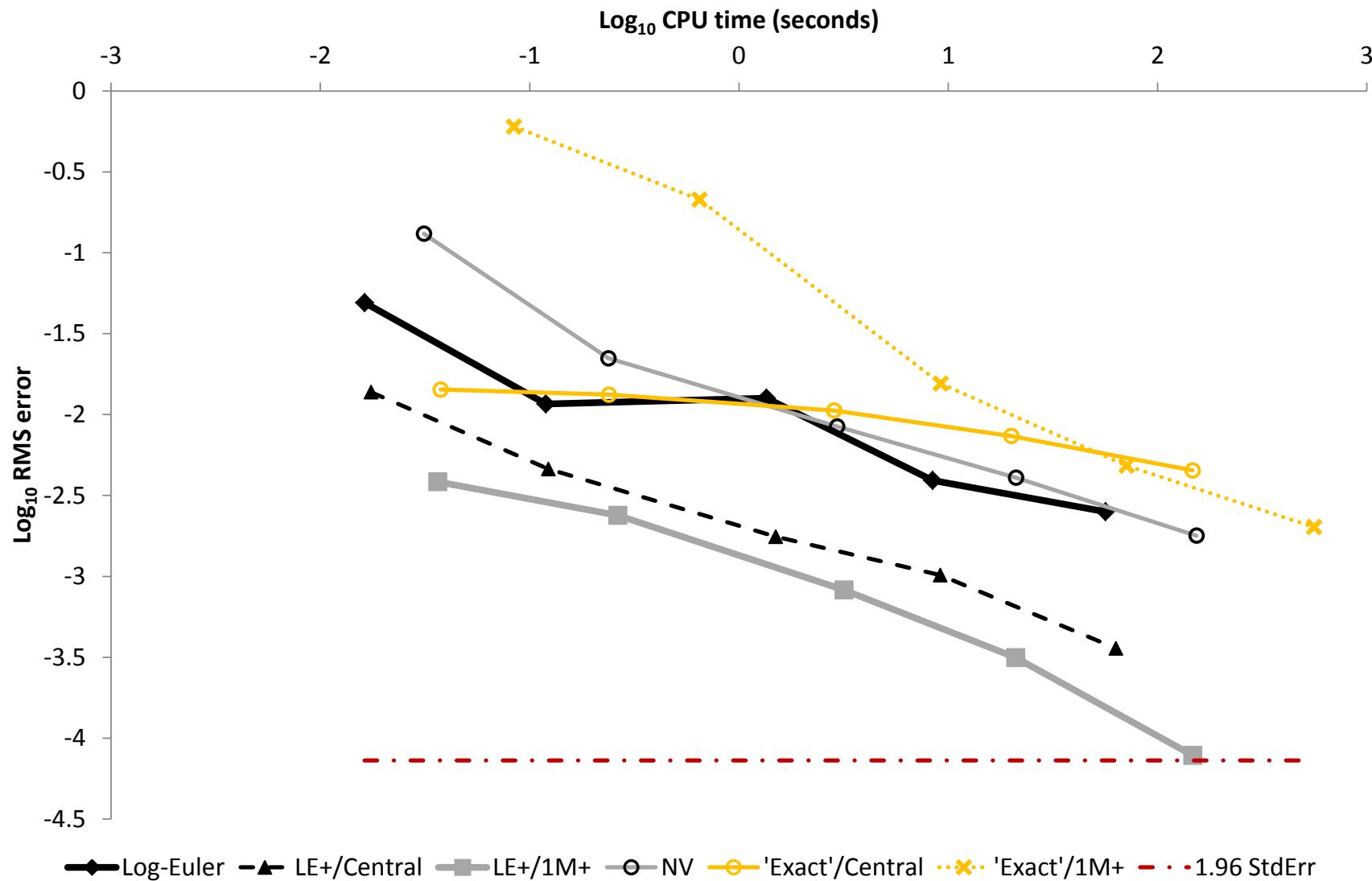
# *SABR $\rho = -60\% - LE+$ /Central (zoom)*



# *SABR $\rho = -60\%$ – Log-Euler+ / 1M+*



# *SABR $\rho = -60\%$ – RMS error*



# *Part V – Fifty shades freed – free SABR*

# Free SABR

Antonov, Konikov and Spector [2015] recently extended the SABR model to the entire real line by allowing the underlying asset to cross the zero boundary, rather than the usual assumption of it being absorbed or reflected. The SDE becomes:

$$dS(t) = \sigma(t)|S(t)|^\beta dW_S(t)$$

$$d\sigma(t) = \alpha\sigma(t)dW_\sigma(t)$$

For the zero correlation case analytical prices (via a two-dimensional integration) are obtained. For non-zero  $\rho$  an approximation is supplied.

# *Free SABR – Islah's approximation*

Most of what we have said so far carries over to the free boundary SABR model, after small adjustments. First of all, Islah's approximation becomes:

$$\frac{dY(t)}{\operatorname{sgn}(Y(t))} = 2\sqrt{|Y(t)|} \sqrt{1-\rho^2} \sigma(t) dZ(t) + \delta(1-\rho^2) \sigma(t)^2 dt$$

once we redefine the transformations as:

$$X(t) = \frac{1}{1-\beta} \operatorname{sgn}(S(t)) \cdot |S(t)|^{1-\beta}$$

$$Y(t) = \operatorname{sgn}(\tilde{X}(t)) \cdot \tilde{X}(t)^2$$

# *Free SABR – Moment matching*

Antonov et al. [2015] show that the PDF of the free boundary Bessel squared (BESQ) process:

$$\frac{dY(t)}{\operatorname{sgn}(Y(t))} = 2\sqrt{|Y(t)|} dW(t) + \delta dt$$

can be expressed in terms of the absorbing and reflecting BESQ process as:

$$p_F(t, y, y_0) = \frac{1}{2} \left( p_R(t, |y|, |y_0|) + \operatorname{sgn}(yy_0) p_A(t, |y|, |y_0|) \right)$$

We have already specified the moment generating function of the absorbing BESQ in earlier slides.

# *Free SABR – Moment matching (2)*

For the reflecting BESQ process, this is given by:

$$\mathbb{E}[Y(t)^a \mid Y(0) = y] = (2t)^a \frac{\Gamma(a + \frac{1}{2}\delta)}{\Gamma(\frac{1}{2}\delta)} \cdot {}_1F_1(-a, \frac{1}{2}\delta, -\frac{y}{2t})$$

From the probability distribution function it follows that the first moment of the asset only depends on the absorbing BESQ moment generating function, whereas the second moment only depends on the reflecting BESQ moment generating function.

Since the free SABR process lives on the real line, we have chosen a normal distribution to match its first two moments.

# *Free SABR – Moment matching (3)*

It should be noted that Antonov et al. [2015] suggest a moment-matching scheme directly on the free BESQ process (without Islah's approximation). They:

- ▶ Moment match a quadratic Gaussian to the BESQ process for sufficiently high values of the asset
- ▶ Invert the free BESQ cdf in other cases

The simulated asset is found by inverting the transformation. We prefer directly matching the moments of the asset. We do not compare to this scheme here, though we expect a full moment-matching approach to be more competitive.

# *Euler schemes for free SABR*

An Euler+ scheme might work, paired of course with an integrated variance scheme:

$$S(t) = S(0) + |S(0)|^\beta \frac{\rho}{\alpha} (\sigma(t) - \sigma(0)) \\ + |S(0)|^\beta \sqrt{1 - \rho^2} \int_0^t \sigma(u) dW_S(u)$$

The log-Euler+ scheme would not be a candidate here, since the purpose of the log-transformation is to preserve positivity.

There is however nothing that prohibits us from using different transformations in different regions.

# *Euler schemes for free SABR (2)*

We could therefore envisage the following hybrid scheme working well:

- ▶ Use  $\ln(S)$  as the transformation for  $S > \varepsilon$ , effectively using the log-Euler+ scheme in this region
- ▶ Use the Euler+ scheme for  $S$  in  $[-\varepsilon, \varepsilon]$
- ▶ Use  $\ln(-S)$  for  $S < -\varepsilon$

We will refer to this scheme as the switching( $\varepsilon$ ) scheme in the remainder. Results will be added to this presentation in due course.

# *Free SABR example*

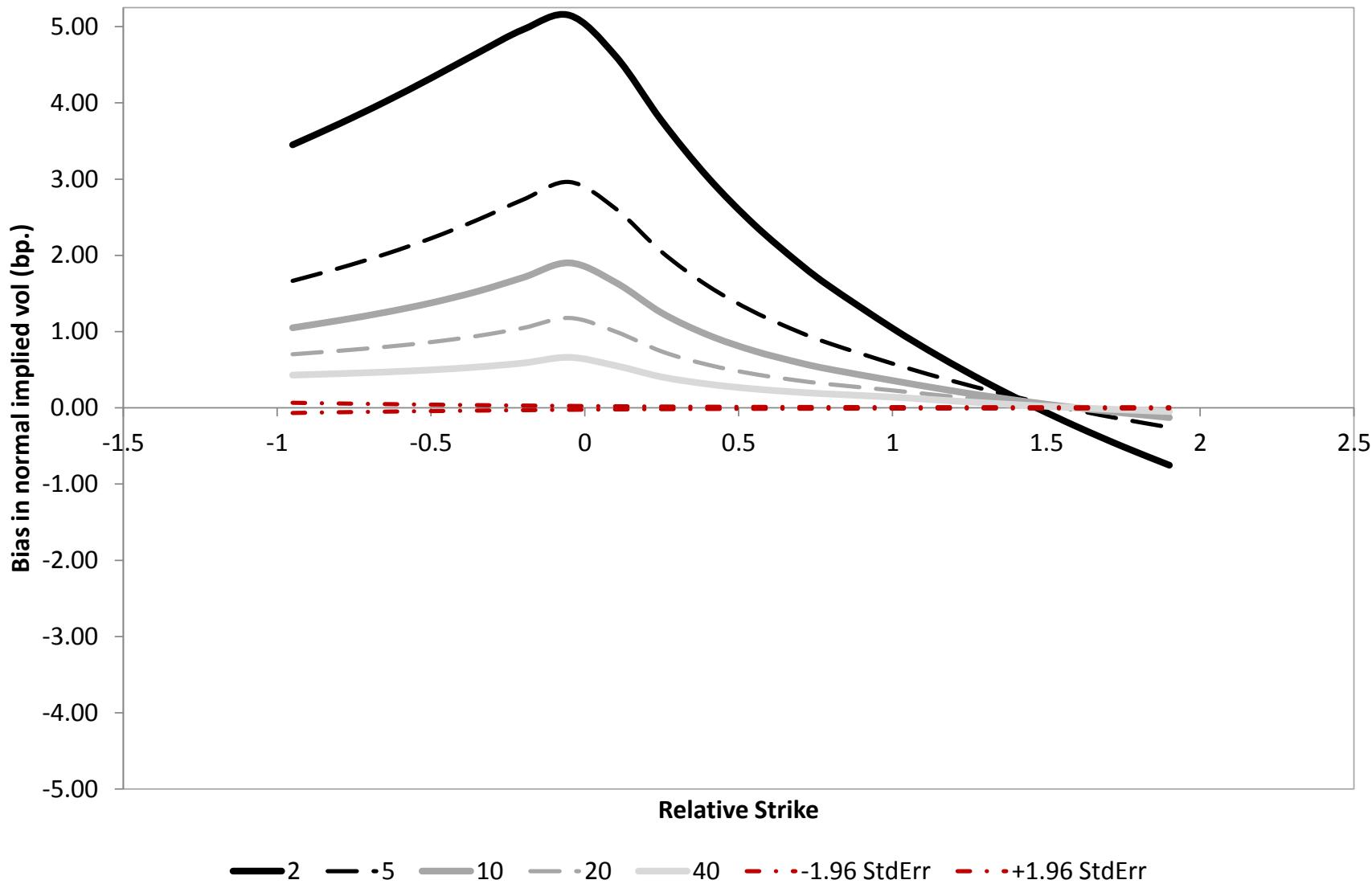
In order to compare the various suggested schemes for the free SABR model, we consider Case I of Antonov et al. [2015]:

$\sigma(0)$	$\alpha$	$\beta$	$\rho$	$S(0)$
$0.6S(0)^{1-\beta}$	30%	25%	-30%	50 bps

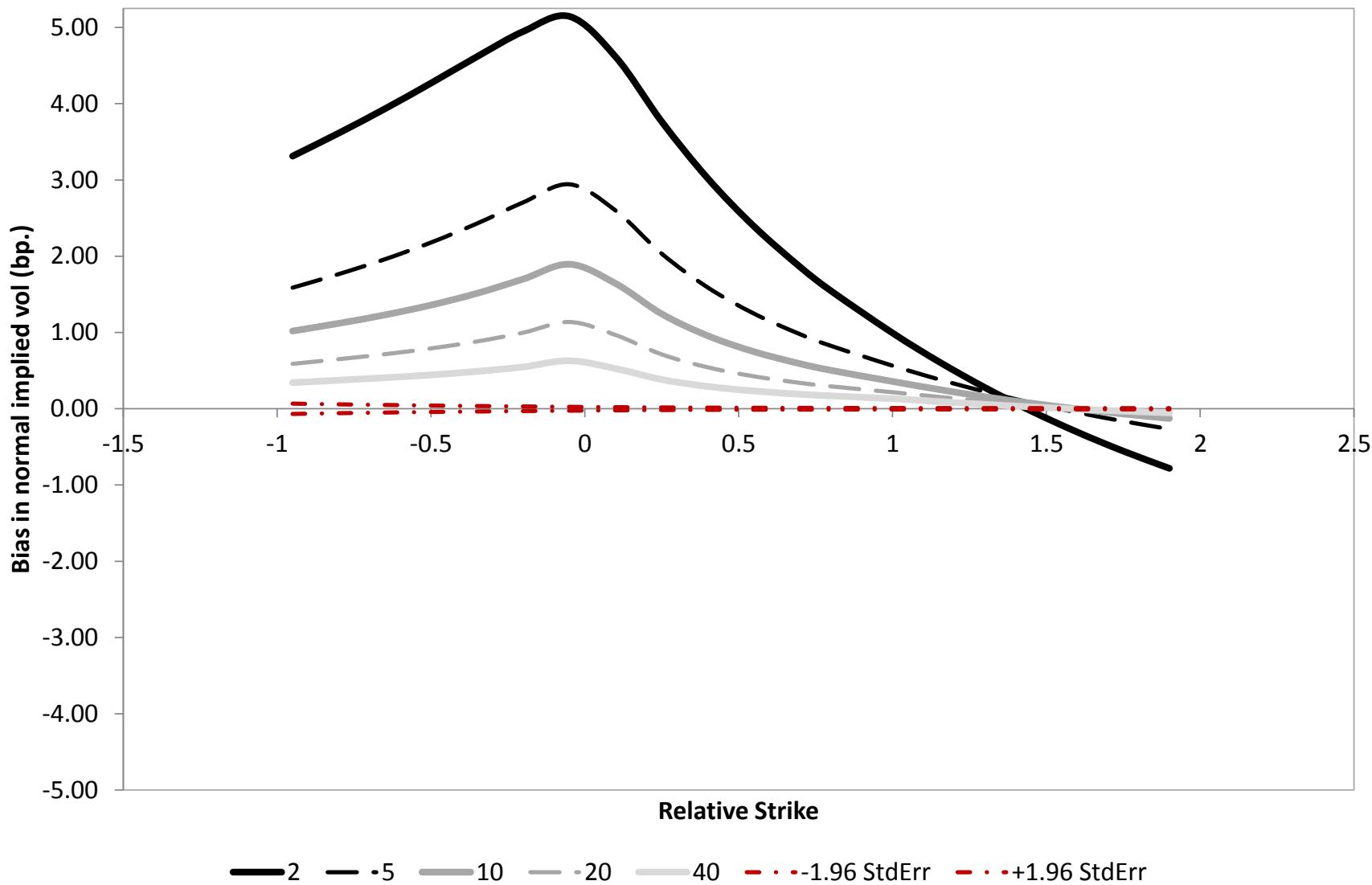
We focus on a 3y European call option.

In the graphs the strike is denoted as a relative strike, i.e.  $K / S(0)$ .

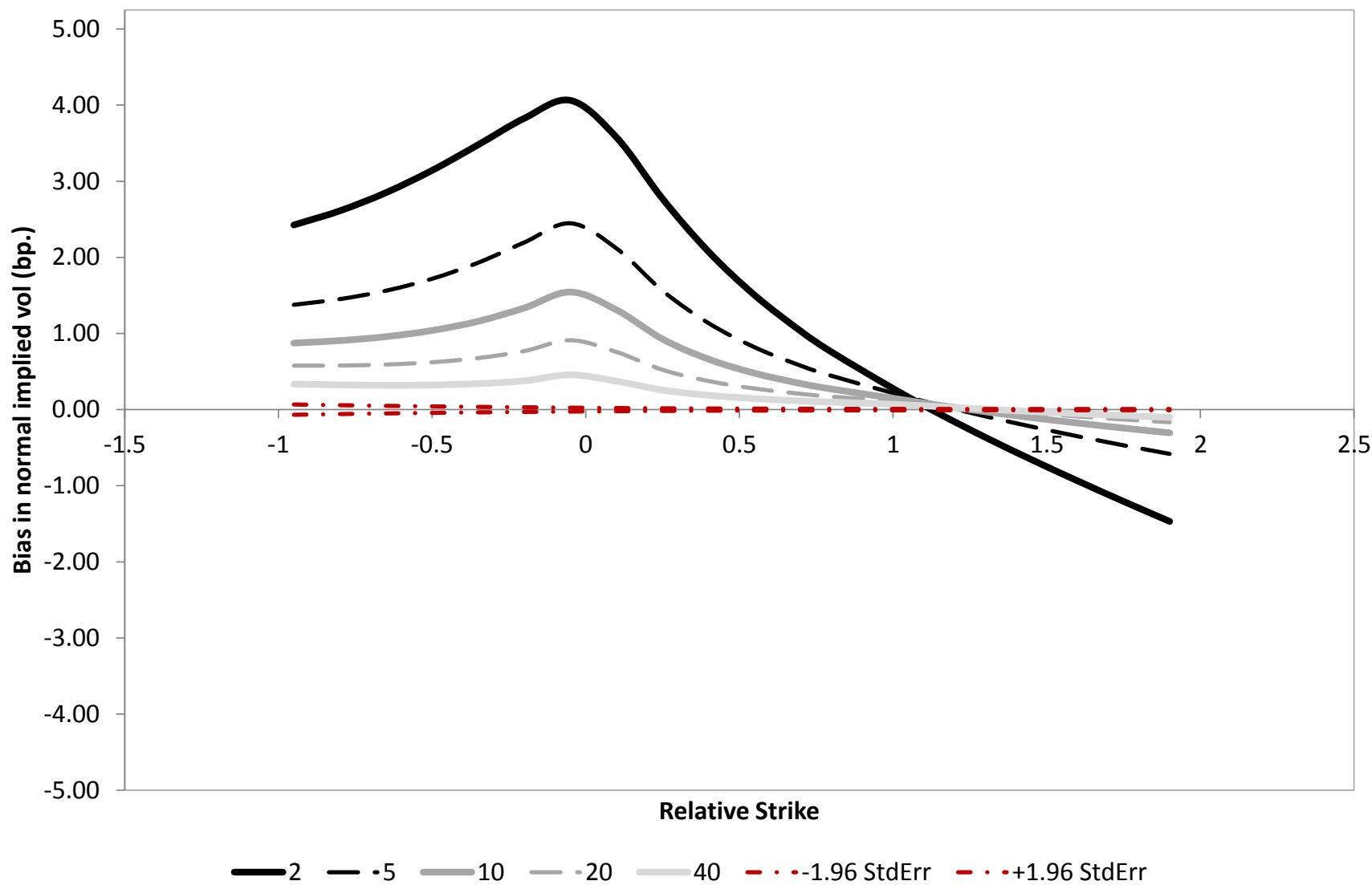
# *Euler+ / Central*



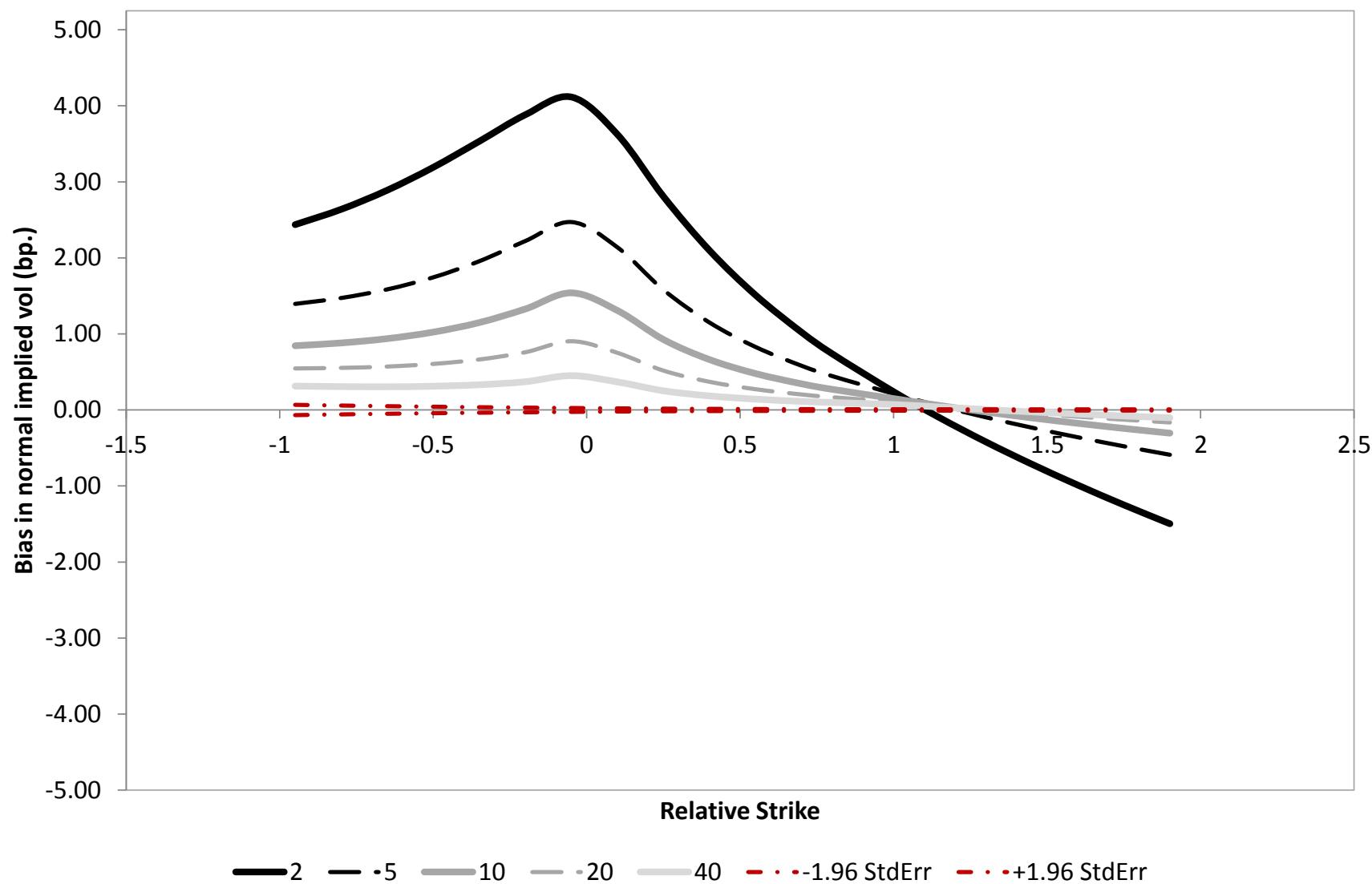
# Euler+ / 1M+



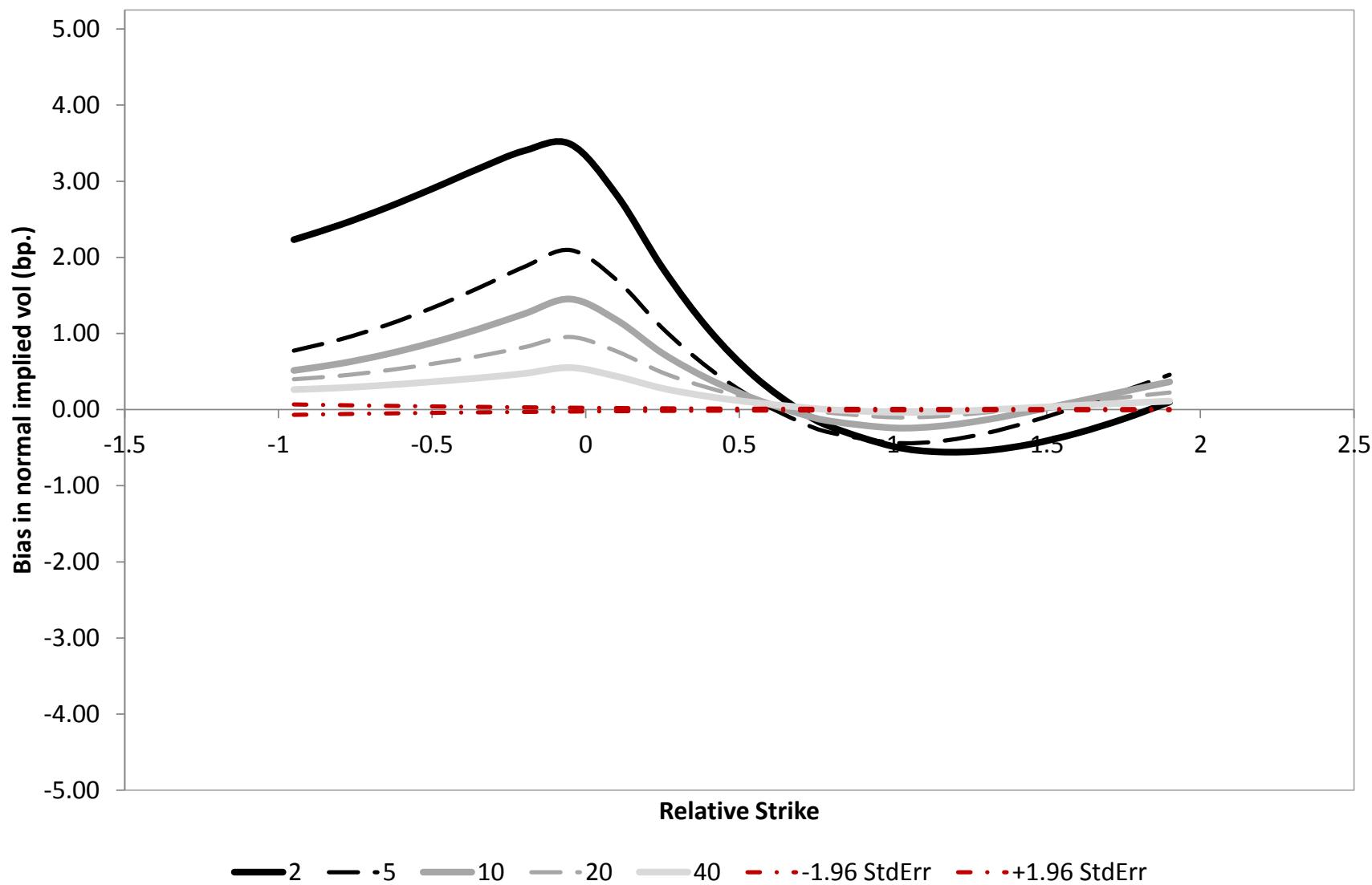
# Moment-matched / Central



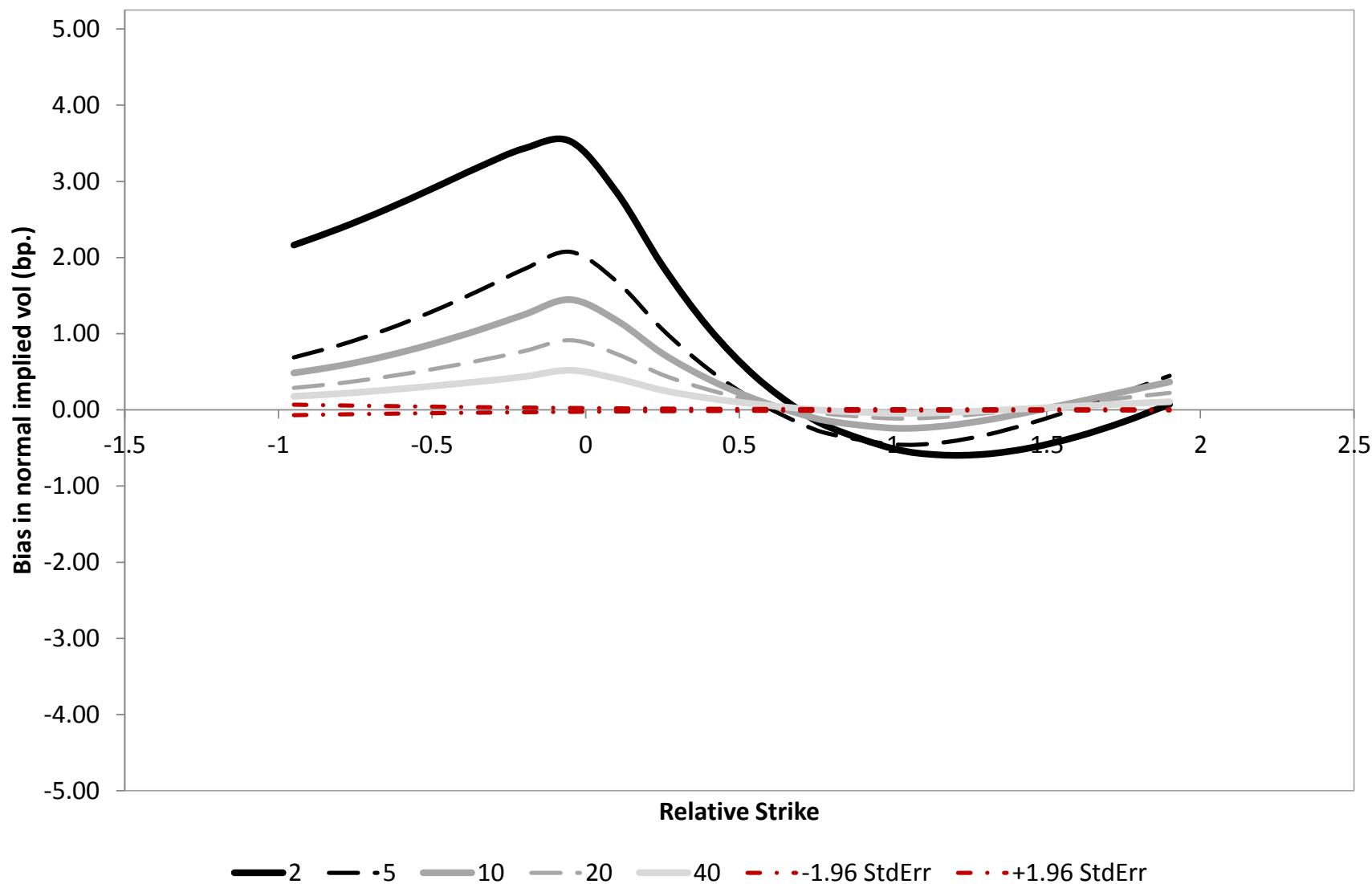
# Moment-matched / 1M+



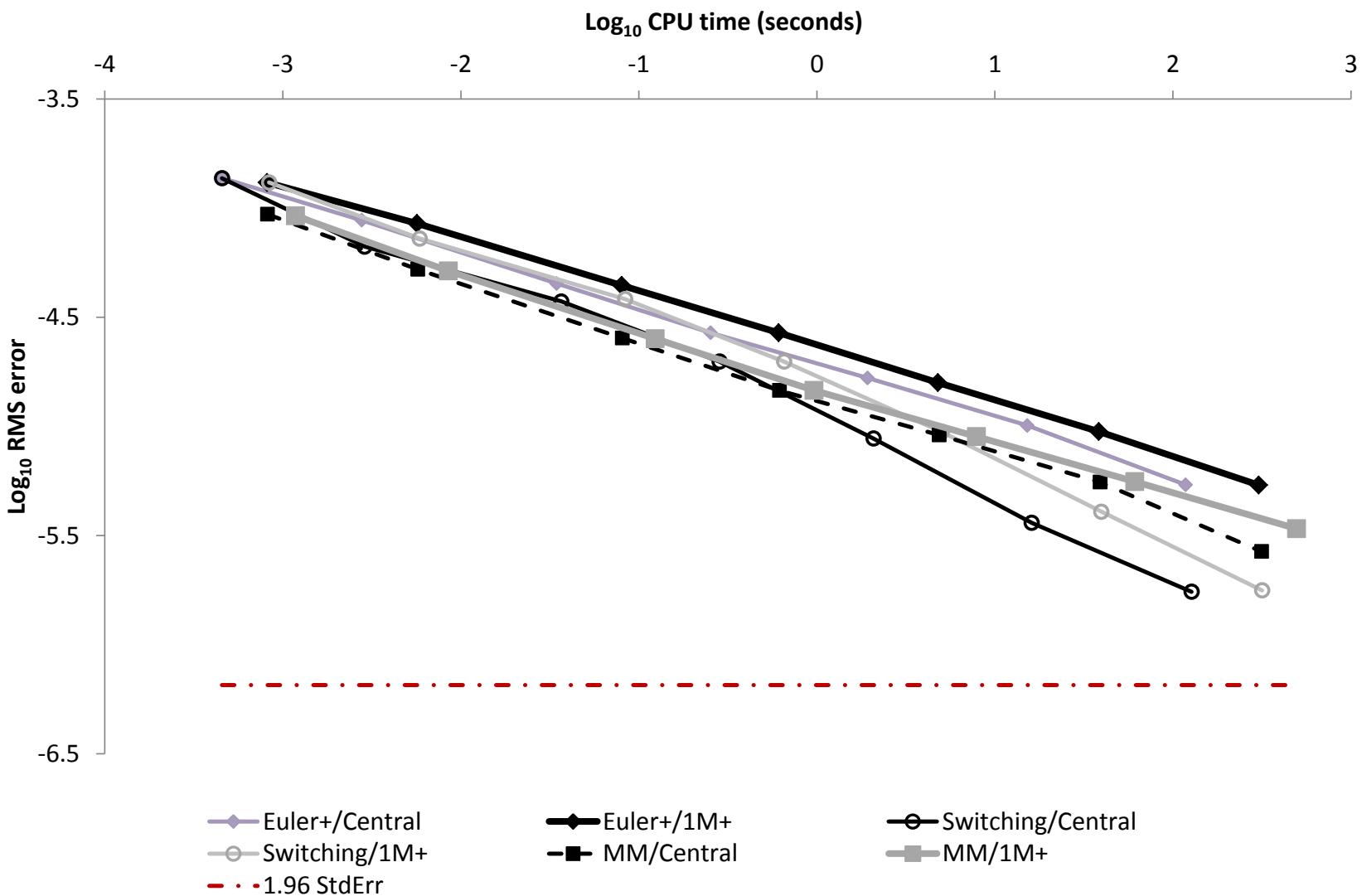
# *Switching(50bps) / Central*



# *Switching(50bps) / 1M+*



# Free SABR – RMS error



# **Conclusions**

## **Approximating the integrated variance process**

- ▶ Can be improved using conditioning schemes
- ▶ Central scheme remains a good workhorse and is competitive when small time steps are required or needed

## **SABR with zero correlation**

- ▶ Long time stepping becomes possible with either a QG approximation or an exact simulation for the asset (slightly more costly in terms of CPU time)

# ***Conclusions (2)***

## **SABR with non-zero correlation**

- ▶ For large correlations, approximation breaks down
- ▶ Log-Euler schemes coupled with an efficient scheme to sample the integrated variance perform best, but long time stepping is impossible

## **Free SABR**

- ▶ Extensions possible to free-boundary SABR
- ▶ Hybrid Euler/Log-Euler scheme, coupled with an integrated variance scheme, performs well

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