

Optimal contours and controls in semi-analytical option pricing

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The material presented here is joint work with Christian Kahl (FINCAD). The presentation can be found here at http://www.rogerlord.com, where the forthcoming paper will also be posted in the near future:

LORD, R. AND C. KAHL (2016). "Optimal contours and controls".

Material

Main material stems from:

ANDERSEN, L.B.G. AND J. ANDREASEN (2002). "Volatile volatilities", *Risk*, December 2002, pp. 163-168.

CARR, P. AND D.B. MADAN (1999). "Option valuation using the Fast Fourier Transform", *Journal of Computational Finance*, vol. 2, no. 4, pp. 61-73.

LORD, R. AND C. KAHL (2007). "Optimal Fourier inversion in semi-analytical option pricing", *Journal of Computational Finance*, vol. 10, no. 4, pp. 1-30.

JOSHI, M. AND C. YANG (2011). "Fourier transforms, option pricing and controls", working paper, available at: http://ssrn.com/abstract=1941464.

Part I – Pricing options via Fourier inversion

Extended characteristic function

We will start with the concept of an extended characteristic function. For $\mathbf{u} \in \mathbb{C}^n$ it is defined as:

$$\phi(\mathbf{u}) = \mathbb{E}[e^{i\mathbf{u}^T \mathbf{X}(T)}]$$

This is only defined within its strip of regularity:

$$\Lambda_{\mathbf{X}} = \left\{ \mathbf{u} \in \mathbb{C}^n \,\middle|\, \varphi\left(i\,\mathrm{Im}(\mathbf{u}) \right) < \infty \right\}$$

In the following we will drop the term extended as we will always deal with the extended characteristic function.

Pricing options via Fourier inversion

Though one can use approaches inspired by Lévy [1925], Gurland [1948] and Gil-Pelaez [1951], to calculate an option price via Fourier inversion of the characteristic function to obtain $N(d_1)$ and $N(d_2)$ terms, it is advisable to use the more efficient approach of Carr and Madan [1999], dating back to Dubner and Abate [1968].

Their idea is to derive a closed-form expression of the Fourier transform of

$$C_\alpha(k) = e^{\alpha k} C(k) = e^{\alpha k} \mathbb{E}[(S(T) - e^k)^+]$$

and invert that.

Pricing options via Fourier inversion (2)

Lewis [2001] and Lee [2004] have since shown that for any α the integral can be considered as a contour integral in the complex plane:

$$C(k) = R_\alpha(k) + \frac{1}{2\pi}\int_{-\infty-i\alpha}^{\infty-i\alpha} e^{-izk}\,\frac{\phi(z-i)}{-z(z-i)}\,dz$$

For this to hold $v - i(\alpha+1)$ must be in the strip of regularity Λ_X, as then:

$$\left|\varphi(v-i(\alpha+1))\right| \le \varphi(-i(\alpha+1)) = \mathbb{E}\left[S(T)^{\alpha+1}\right] < \infty$$

This means we must choose $\alpha \in (\zeta_- - 1, \zeta_+ - 1)$, where (ζ_-, ζ_+) is the range of allowed moments.

Pricing options via Fourier inversion (3)

The integral has poles at z = i and z = 0 (α = -1 or 0), so when we cross a pole, we have to correct for its residue. Similarly, passing through one halves the contribution. Cauchy's residue theorem states:

$$R_\alpha(k) = \begin{cases} F - K & \alpha < -1 & \text{Put} \\ F - \frac{1}{2}K & \alpha = -1 & \text{Half} - \text{cash secured put} \\ F & -1 < \alpha < 0 & \text{Covered call} \\ \frac{1}{2}F & \alpha = 0 & \text{Half - covered call} \\ 0 & \alpha > 0 & \text{Call} \end{cases}$$

Pricing options via Fourier inversion (4)

Carr-Madan's approach is very convenient as:

- Only one numerical integration is required, integrand decays faster than with Gil-Pelaez;
- α can be used to control the numerical accuracy, as in Lee [2004] and Lord and Kahl [2007];
- Characteristic functions are often known in closed-form for many exponentially affine models;
- Densities of these models are typically not known in closed-form, or, when they are, may involve special functions;

Part II – Optimal contour

Optimal contour

The question is: which contour α should we choose? The initial literature had varying suggestions:

- Carr and Madan [1999]: $\alpha = \alpha_{max}/4$;
- Raible [2000]: $\alpha = 25$;
- Lewis [2001]: $\alpha = -1/2$;
- Schoutens, Simons and Tistaert [2003]: $\alpha = 0.75$:
- Lee [2004]: in the context of a DFT, joint minimisation over α and spacing to minimise the truncation error plus the discretisation error.

Optimal contour (2)

Before we look at the practicalities, the first that springs to mind is:

Is the right choice of α really that important?

To answer this question, we will look at some numerical examples within the Heston stochastic volatility model:

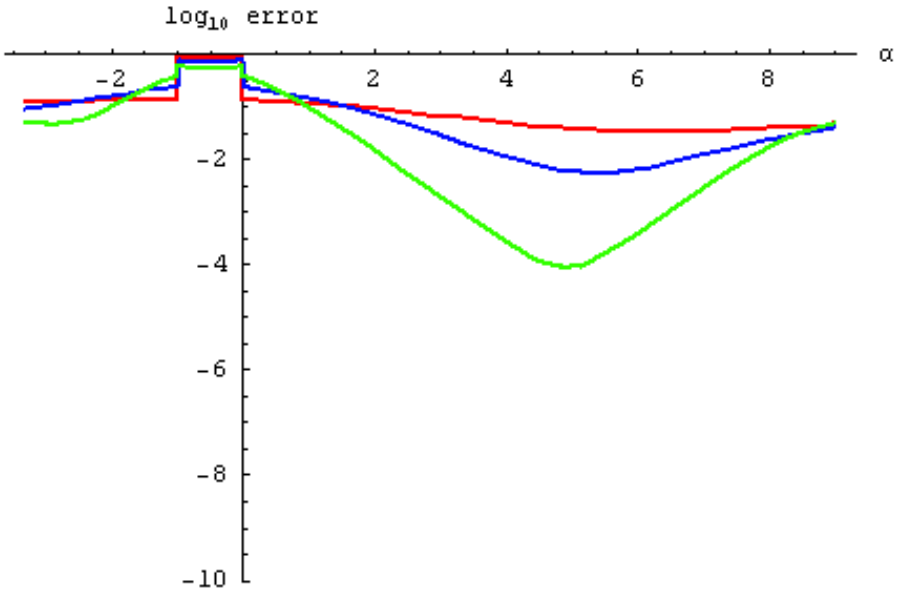
$$dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}\,S(t)dW_S(t)$$

$$dv(t) = -\kappa\big(v(t) - \theta\big)dt + \omega\sqrt{v(t)}dW_v(t)$$

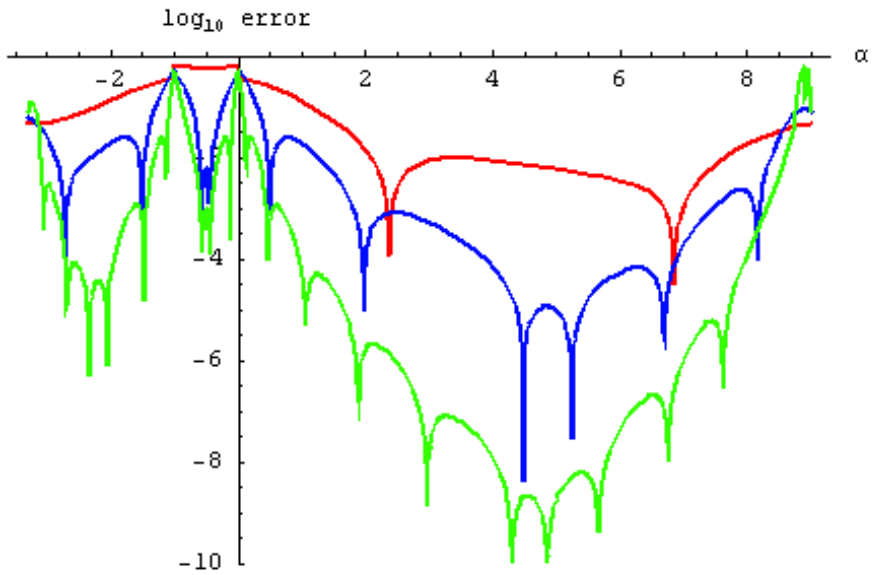
Optimal contour (3)

$F = K = 1$, $\kappa = \omega = 1$, $\theta = v(0) = 0.1$, $\rho = -0.7$, $T = 1y$

Discretisation from Lee [2004]



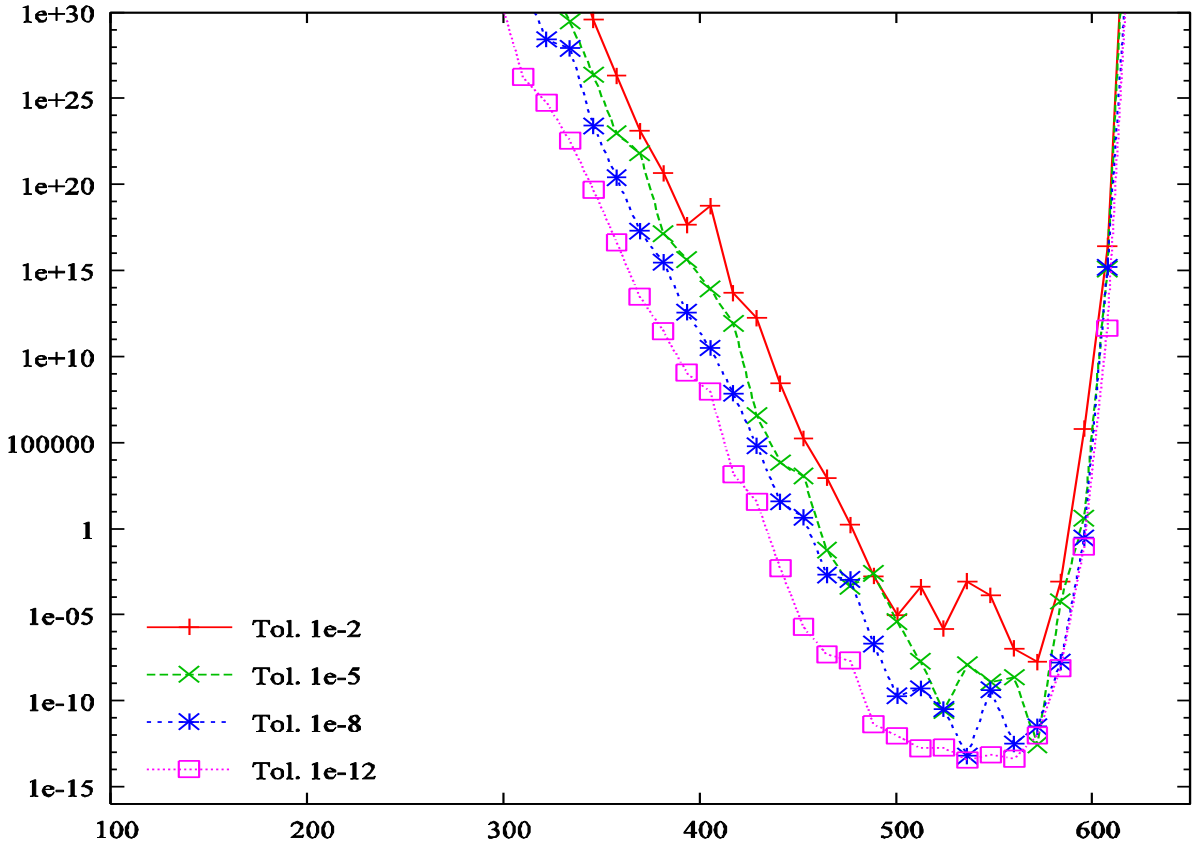
Gauss-Legendre quadratures



Certainly can be beneficial to use the right α...

Optimal contour (4)

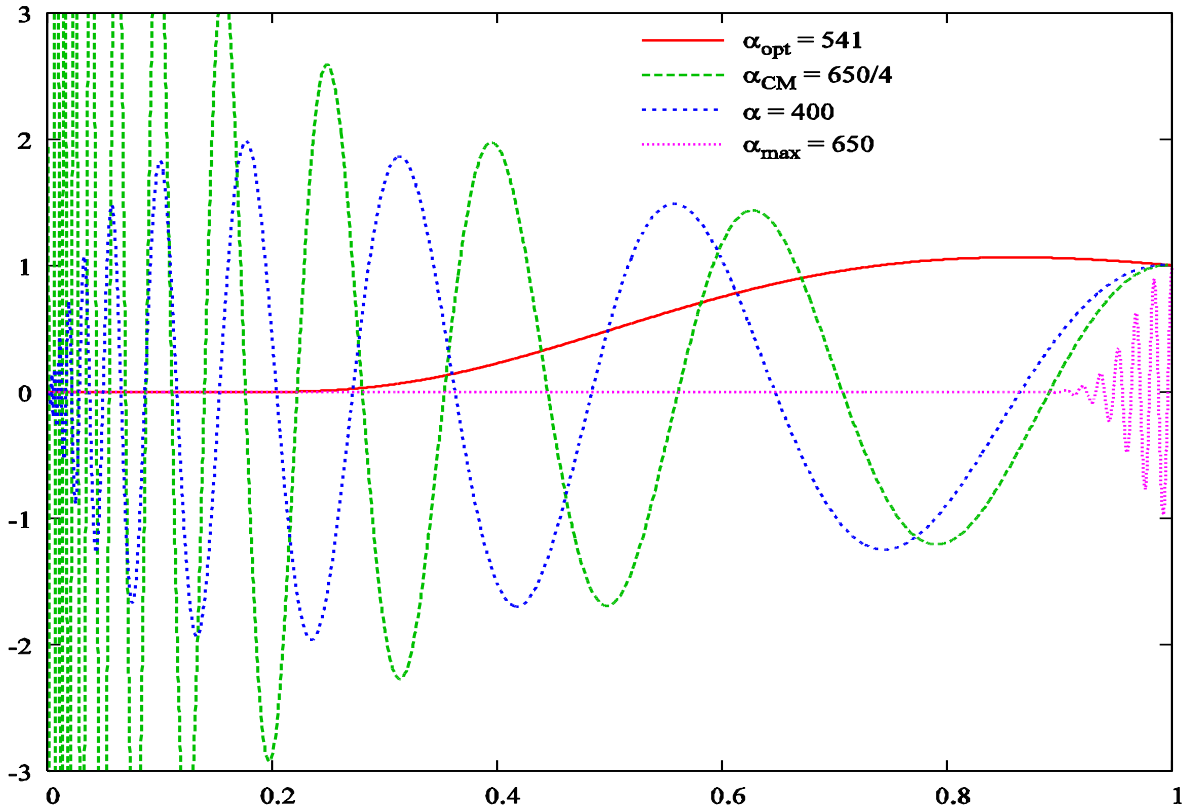
$F = 1, K = 2, \kappa = \omega = 1, \theta = v(0) = 0.1, \rho = -0.9, T = 1w$



... and sometimes it is crucial.

Optimal contour (5)

In the last example, consider the shape of the transformed integrand for various choices of α:



Optimal contour (6)

Some observations:

- Too big or too small α causes cancellation errors;
- Ideally we would like an integrand which is as constant as possible;
- There is a range of α's that yield a similar accuracy.

The last observation at least gives some hope for a heuristic algorithm – we do not have to get the value of the contour exactly right.

Optimal contour (7)

In the remainder we will denote the integrand of the Carr-Madan representation, by:

$$\psi(v,\alpha) = \mathrm{Re}\left(\frac{e^{-ivk}\varphi(v - i(\alpha+1))}{-(v - i\alpha)(v - i(\alpha+1))} \right)$$

The degree of oscillation of a function can be measured by the total variation (TV). For a function $f: \mathbb{R} \to \mathbb{R}$ on [-1, 1] this equals:

$$TV(f) = \int_{-1}^{1} \left| \frac{\partial f(x)}{\partial x} \right| dx$$

Optimal contour (8)

As shown by Förster and Petras [1991], a small total variation ensures that the approximation error of an applied numerical integration scheme is reduced.

We should choose α such that the TV is minimised:

$$\alpha^* = \underset{\alpha \in (\alpha_{Min}, \alpha_{Max})}{\arg\min} \; e^{-\alpha k} \int_0^\infty \left| \frac{\partial}{\partial v} \psi(v, \alpha) \right| dv$$

Clearly this optimisation procedure is of a theoretical nature, as it would require many more function evaluations than the original problem. We can improve on this however.

Optimal contour (9)

We notice that the integrand attains its maximum value in v = 0. Moreover, when the integrand is monotone in v, the TV equals:

$$e^{-\alpha k}\int_0^{\infty}\left|\frac{\partial}{\partial v}\psi(v,\alpha)\right|dv=\left|e^{-\alpha k}\psi(0,\alpha)\right|$$

since the characteristic function vanishes at infinity. In this case, the following problem has the same first order condition for optimality:

$$\alpha^{*}=\underset{\alpha\in(\alpha_{Min},\alpha_{Max})}{\arg\min}\left|C(k,\alpha)-C(k)\right|$$

Optimal contour (10)

Therefore, we arrive at the following problem:

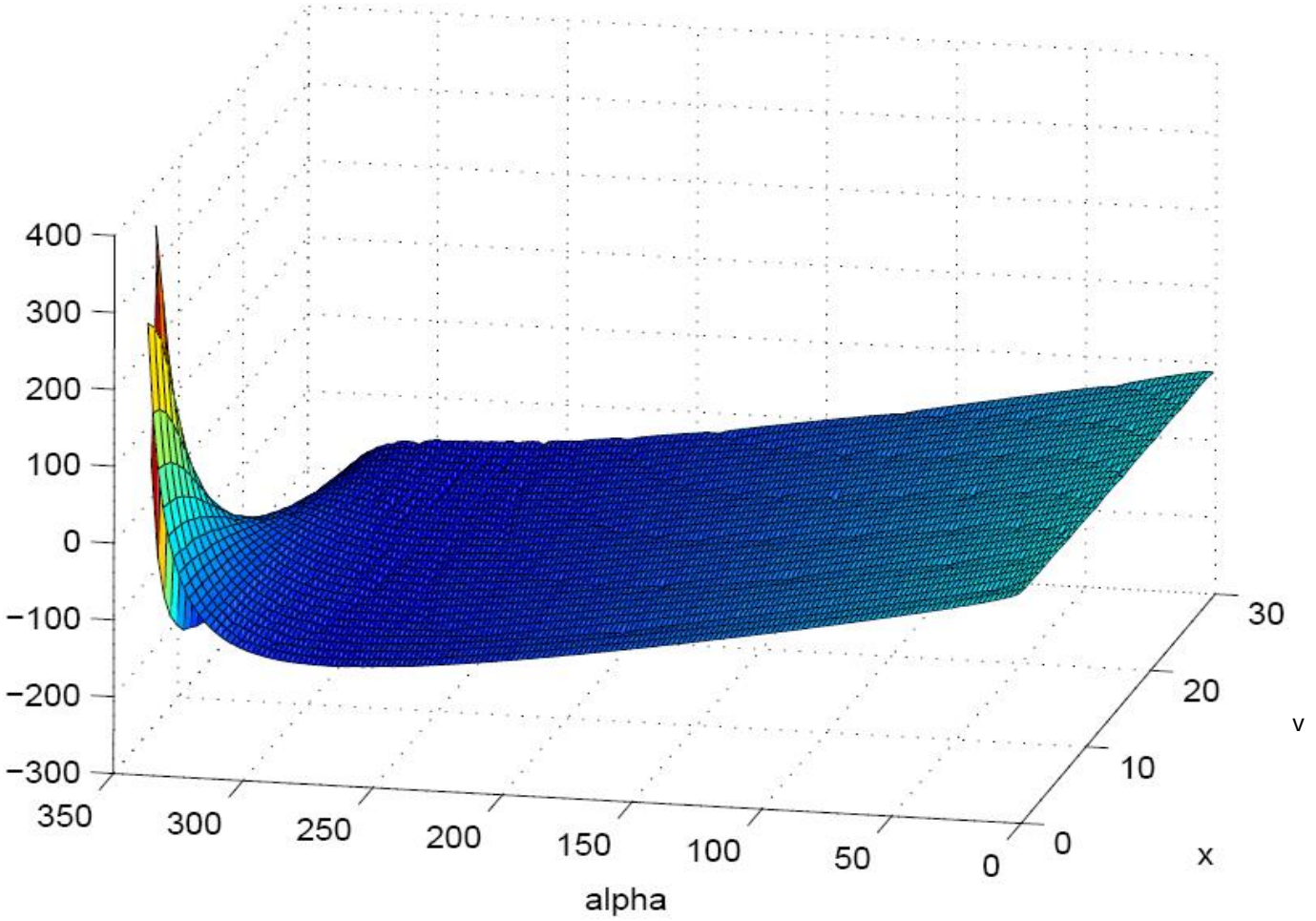
$$\alpha^* = \underset{\alpha \in (\alpha_{\text{Min}}, \alpha_{\text{Max}})}{\arg\min} \left| e^{-\alpha k} \psi(0,\alpha) \right|$$
$$= \underset{\alpha \in (\alpha_{\text{Min}}, \alpha_{\text{Max}})}{\arg\min} -\alpha k + \tfrac{1}{2} \ln\left(\psi(0,\alpha)^2 \right) \equiv \Psi(\alpha, k)$$

A payoff-independent alternative, with closed-form solutions for the Black-Scholes and Variance Gamma models, can be obtained by ignoring the payoff transform, see also Lord and Kahl [2007]:

$$\alpha^* = \underset{\alpha \in (\alpha_{\text{Min}}, \alpha_{\text{Max}})}{\arg\min} -\alpha k + \ln\left(\phi(0,\alpha) \right) \equiv \Phi(\alpha, k)$$

Optimal contour (11)

The integrand is typically minimised overall:



Optimal contour (12)

Daniels [1954] has proven that the characteristic exponent has a unique global minimum. In our case, $\Psi(\alpha, k)$ has three local minima:

- $\alpha_1 \in (0, \alpha_{max})$ (call);
- $\alpha_2 \in (-1,0)$ (covered call);
- $\alpha_3 \in (\alpha_{min}, -1)$ (put).

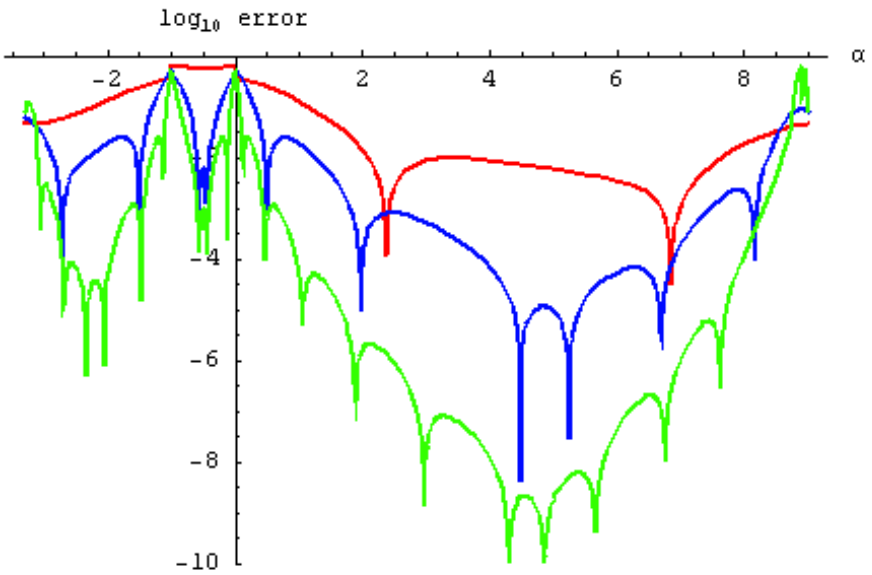
As a rule of thumb, we recommend a heuristic:

- When $F > K$, price the put ($\alpha < 0$);
- When $F < K$, price the call ($\alpha > 0$)

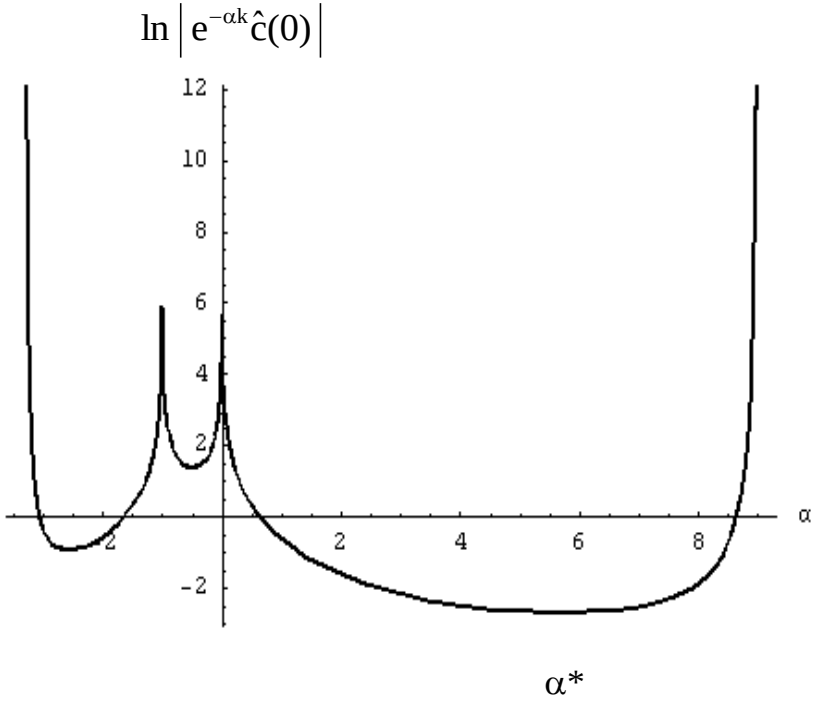
Optimal contour (13)

$F = K = 1$, $\kappa = \omega = 1$, $\theta = v(0) = 0.1$, $\rho = -0.7$, $T = 1y$

Gauss-Legendre quadratures



Location of our α*

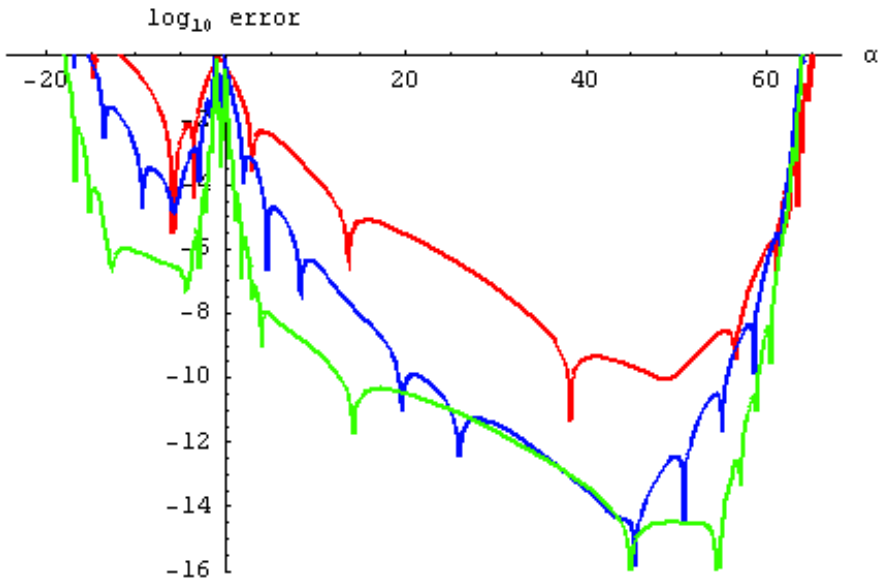


Same example as on first slides, ATM

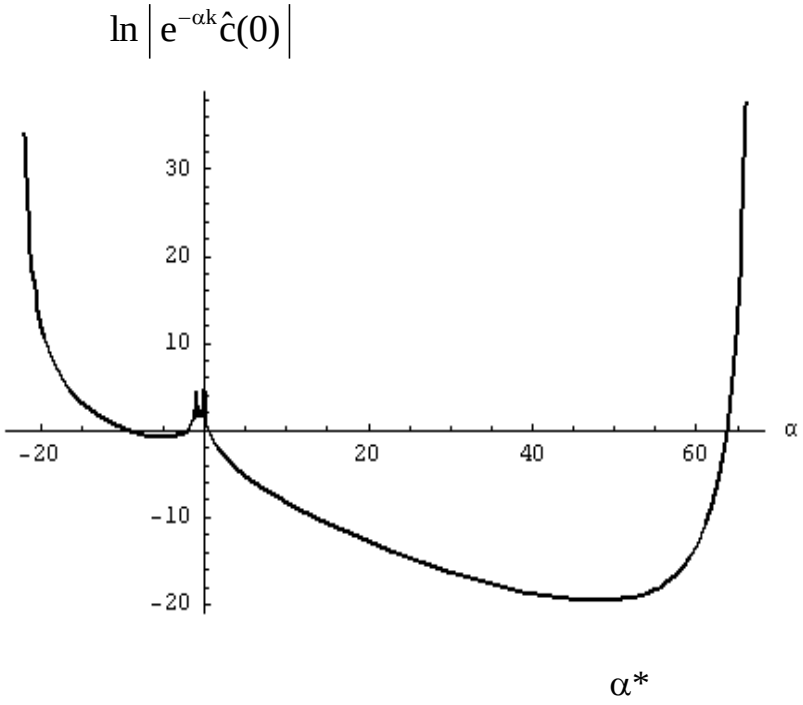
Optimal contour (14)

$F = 1, K = 1.5, \kappa = \omega = 1, \theta = v(0) = 0.1, \rho = -0.7, T = 0.1y$

Gauss-Legendre quadratures



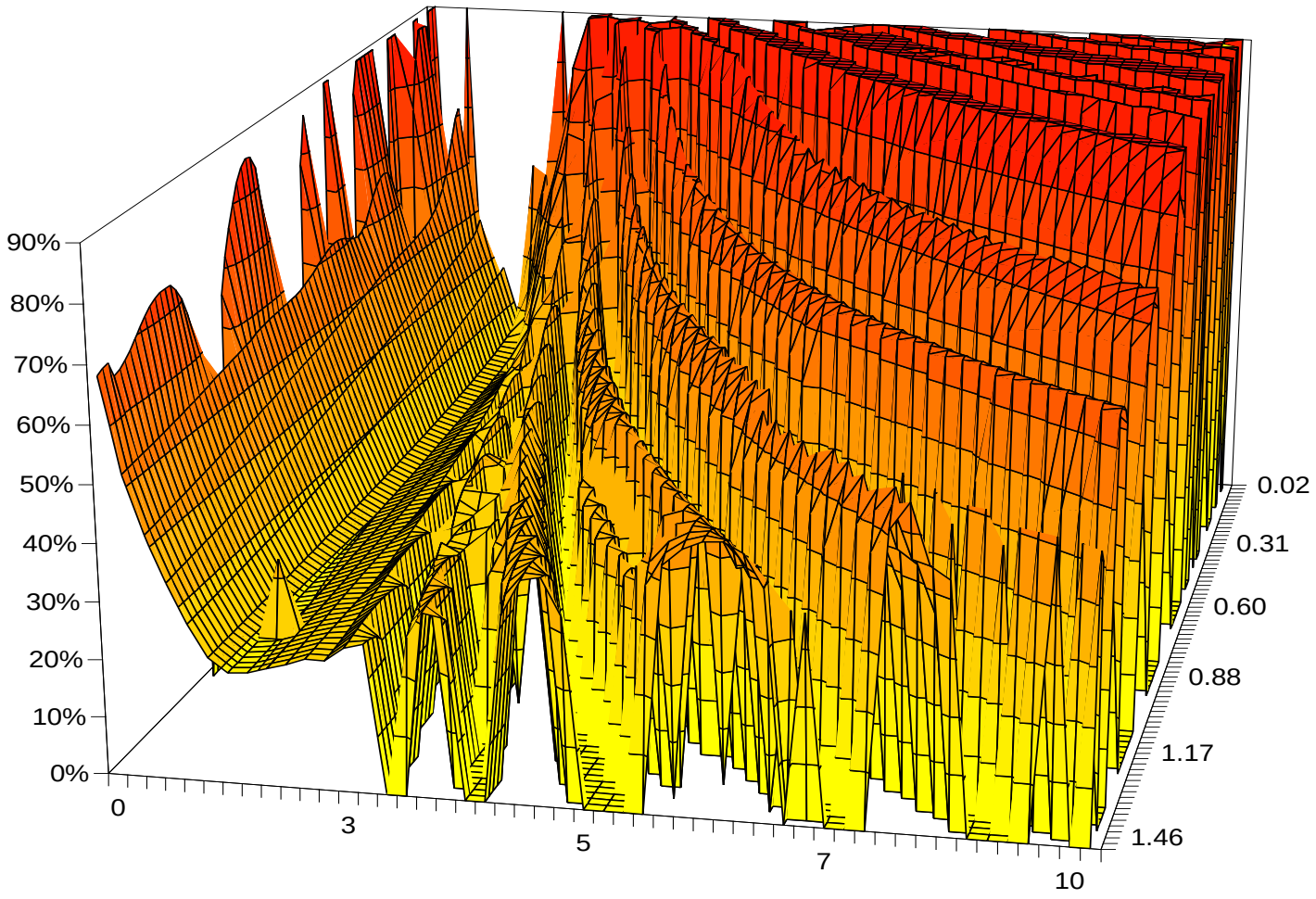
Location of our α*



As previous slide, but OTM and short maturity

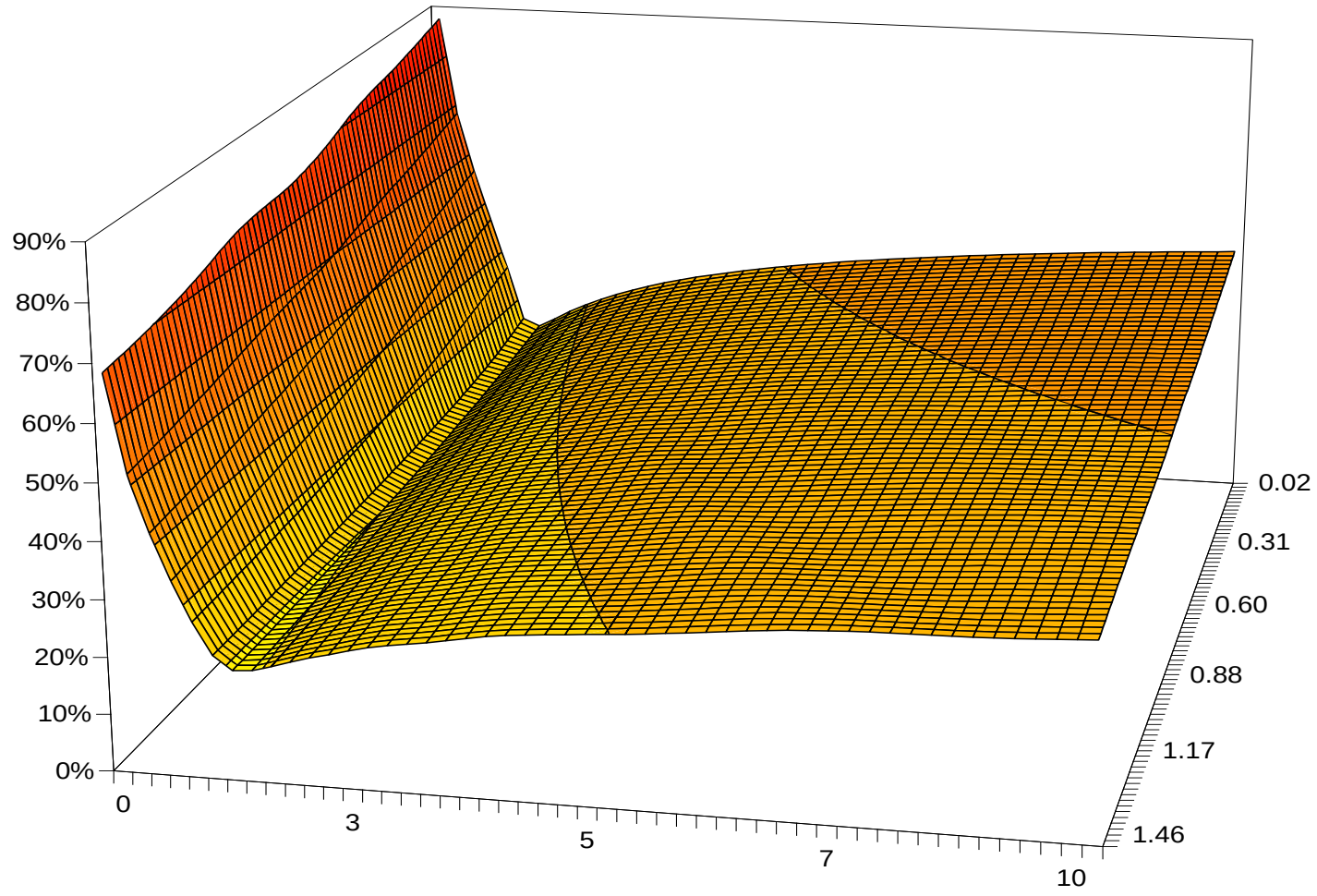
Optimal contour (15)

Vol surface with max. 35 abscissae, TOL = 10^{-3}, α = -1/2



Optimal contour (16)

With the optimal α 12 abscissae are sufficient



Optimal contour (17)

Our choice of damping coefficient coincides with that of Choudhury and Whitt [1997] for non-probability transforms. The underlying idea comes from saddle point approximations, pioneered by Daniels [1954] in the context of calculating probability densities:

$$p(x) = \frac{1}{2\pi i} \int_{-\infty - i\alpha}^{\infty - i\alpha} e^{-izx} \phi(z)\, dz$$

Here, α is chosen as $\alpha^* = \arg \min_{\alpha \in (\alpha_-, \alpha_+)} e^{-\alpha k} \phi(-i\alpha)$.

Saddle point methods typically continue by expanding ϕ around the saddle point to obtain approximations for cdfs. See also Rogers and Zane [1998].

Optimal contour (18)

To wrap up our findings on the optimal contour:

- We have demonstrated the need for an optimal α;
- We recommend to combine it with a transformation to the unit interval and adaptive quadratures;
- Optimal α approximated via a univariate minimisation; minimal extra cost when calculating a volatility surface;
- Large efficiency gain, in particular for short maturities and OTM / ITM options.

Part III – Control variates

Control variates

Andersen and Andreasen [2002] in their paper suggested an approach that can be used as an alternative to finding an optimal contour. The idea behind the approach is exactly the same as the control variate approach used in Monte Carlo algorithms.

Using a Black-Scholes control leads to:

$$C(k) = \frac{e^{-\alpha k}}{\pi} \mathrm{Re} \int_0^\infty e^{-ivk} \frac{\phi\big(v - i(\alpha+1)\big) - \phi_{BS}\big(v - i(\alpha+1)\big)}{-\big(v - i\alpha\big)\big(v - i(\alpha+1)\big)} dv + BS(k, \sigma)$$

The more the control is like the model, the better it is.

Control variates (2)

The forwards are to be set equal. Note that this representation is valid for any choice of the damping coefficient α within the strip of regularity of the model, as the integrand no longer has any poles at $z = 0$ or $z = i$, since:

- The forwards are obviously equal: $\phi(-i) = \phi_{BS}(-i)$
- In $z = i$ we also have $\phi(0) = \phi_{BS}(0) = 1$

Second, for numerical stability we can use the age-old adage "Price the out-of-the-money option", and use put-call parity to arrive at the desired option price.

Control variates (3)

We so far have not specified how to choose the volatility parameter σ. In the Heston model, Andersen and Andreasen [2002] used the Black-Scholes model that follows from the Heston model by setting the vol of variance to 0, as a control variate.

For the remainder, let us assume for ease of exposure:

- Call and put prices are homogeneous of order 1
- The forward equals 1

Control variates (4)

Joshi and Yang distinguish three cases in determining how to choose σ:

- **$\alpha = 0$ or $\alpha = -1$:** The numerator in $v = 0$ already equals zero. Therefore they choose σ such that the first derivative of the numerator is zero. For $\alpha = 0$:

$$\phi'(-i) = \phi'_{BS}(-i) \Leftrightarrow \sigma^2\tau = 2\,\mathrm{Im}\left(\phi'(-i)\right)$$

and for $\alpha = -1$:

$$\phi'(0) = \phi'_{BS}(0) \Leftrightarrow \sigma^2\tau = -2\,\mathrm{Im}\left(\phi'(0)\right)$$

Control variates (5)

- **$\alpha \neq 0$ and $\alpha \neq 1$:** Joshi and Yang ensure the two characteristic functions agree in v = 0:

$$\phi\big(-i(\alpha+1)\big) = \phi_{BS}\big(-i(\alpha+1)\big)$$

which can be achieved by setting:

$$\sigma^2\tau = \frac{2\ln\big(\phi\big(-i(\alpha+1)\big)\big)}{\alpha(\alpha+1)}$$

Essentially all these choices are a form of moment-matching of the Black-Scholes model to our other, as yet unspecified, model.

Control variates (6)

Focussing on the behaviour of the integrand in zero is very similar to how we earlier chose the contour.

For the choice of contour Joshi and Yang argue that the choice should be between $\alpha = -1/2$ (like Lewis) or $\alpha = 0$. Their numerical tests seem to suggest $\alpha = 0$.

Part IV – Contour and control

Control and contour

It is interesting to compare the performance of the control to making a smart choice of the contour. In addition, there might be additional benefits by combining the two algorithms.

Since the above comparison has not been made in Joshi and Yang's paper, we will look into this for both the Variance Gamma model, and the Heston model.

We will investigate the relative performance of Joshi and Yang's control (with $\alpha = 0$) against using the payoff-dependent α. Moreover, we will also combine the two approaches.

Control and contour (2)

The test case for the Variance Gamma model is:

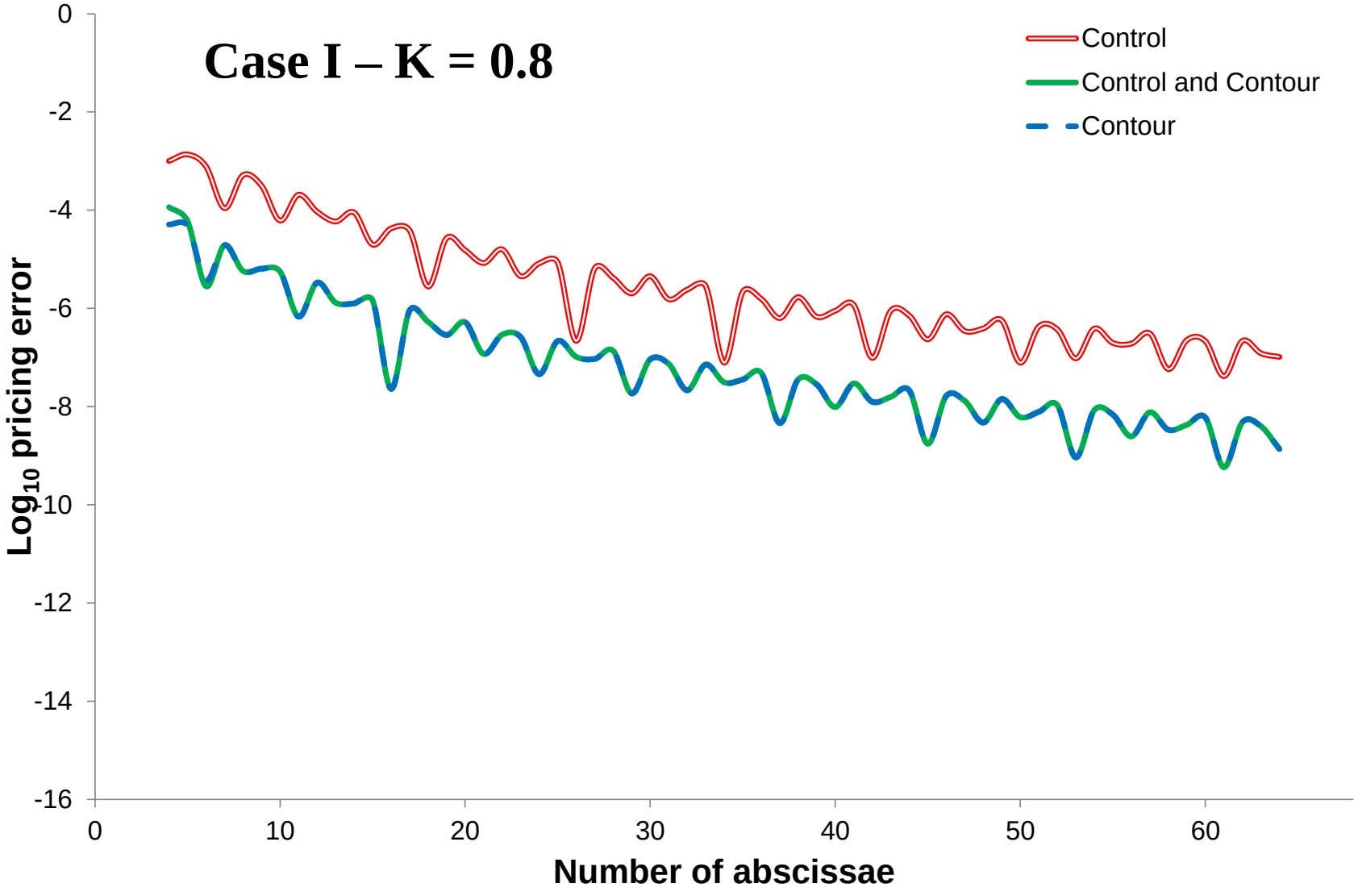
Case	σ	θ	ν	T
I	0.12	-0.33	0.16	0.25

For the Heston model, we take the following test case:

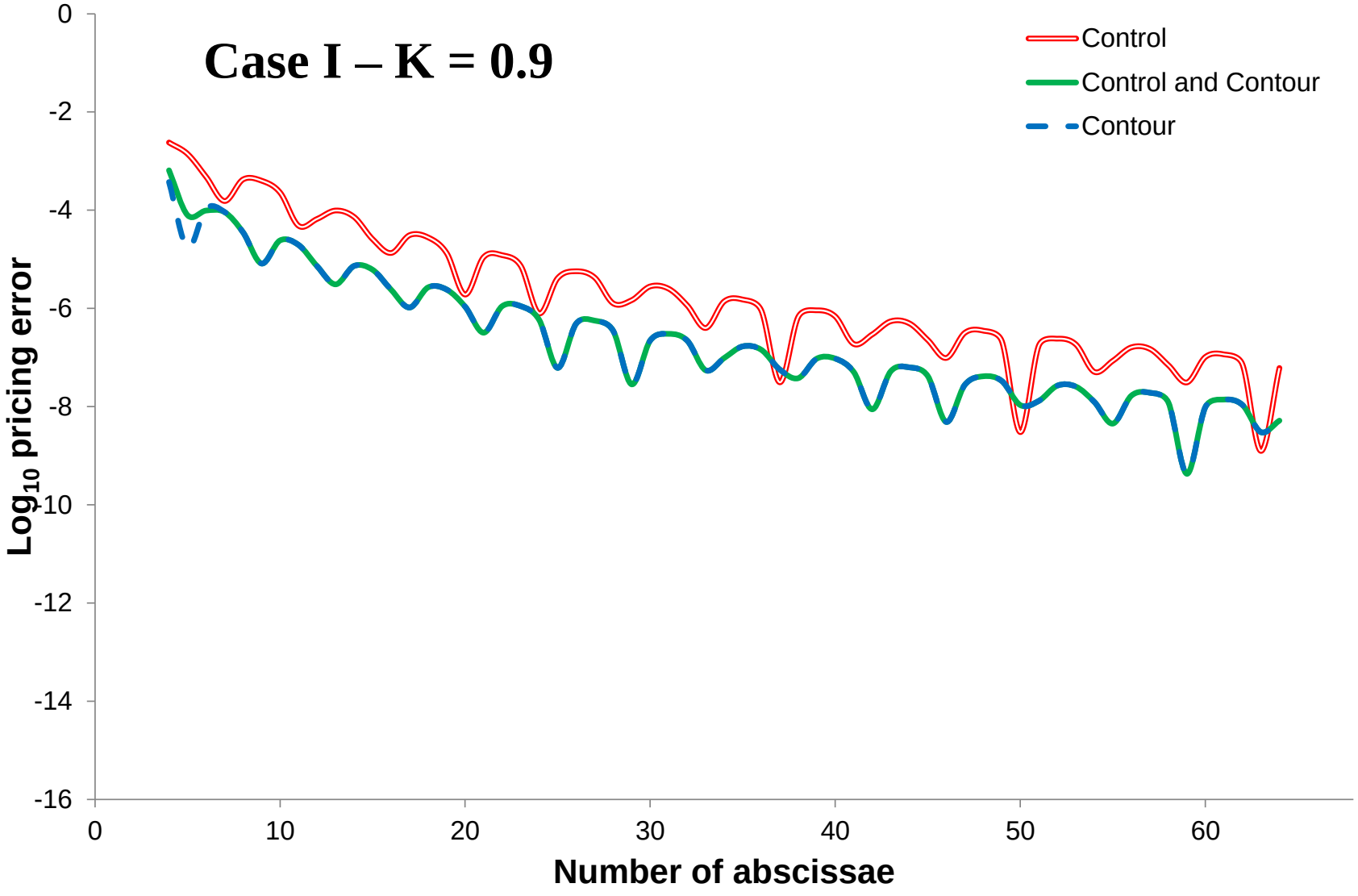
Case	κ	θ	ω	ρ	v(0)	T
II	0.5	0.04	1.0	-0.9	0.04	1/12

We use Gauss-Laguerre quadratures in both cases.

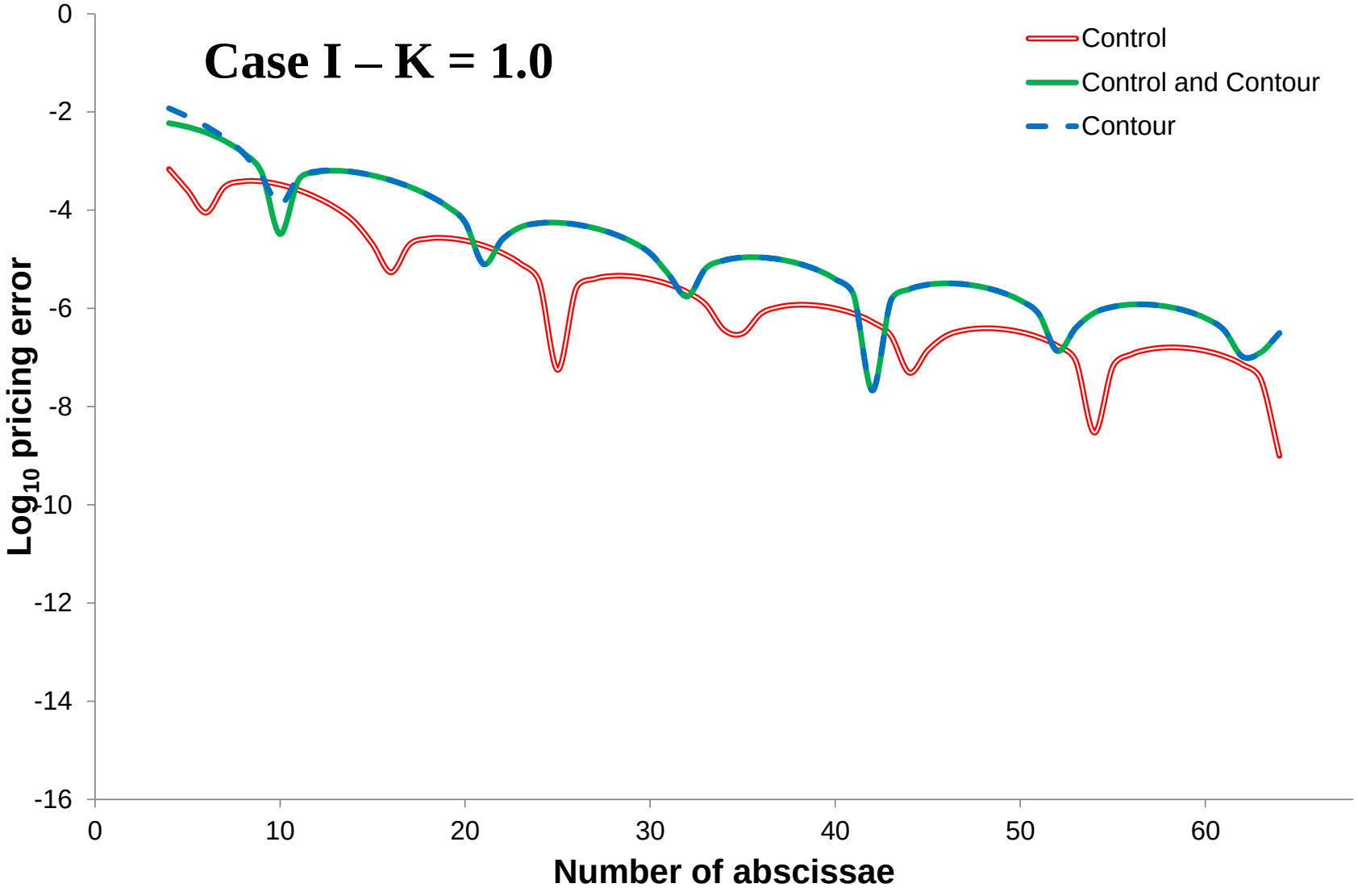
Control and contour (3)



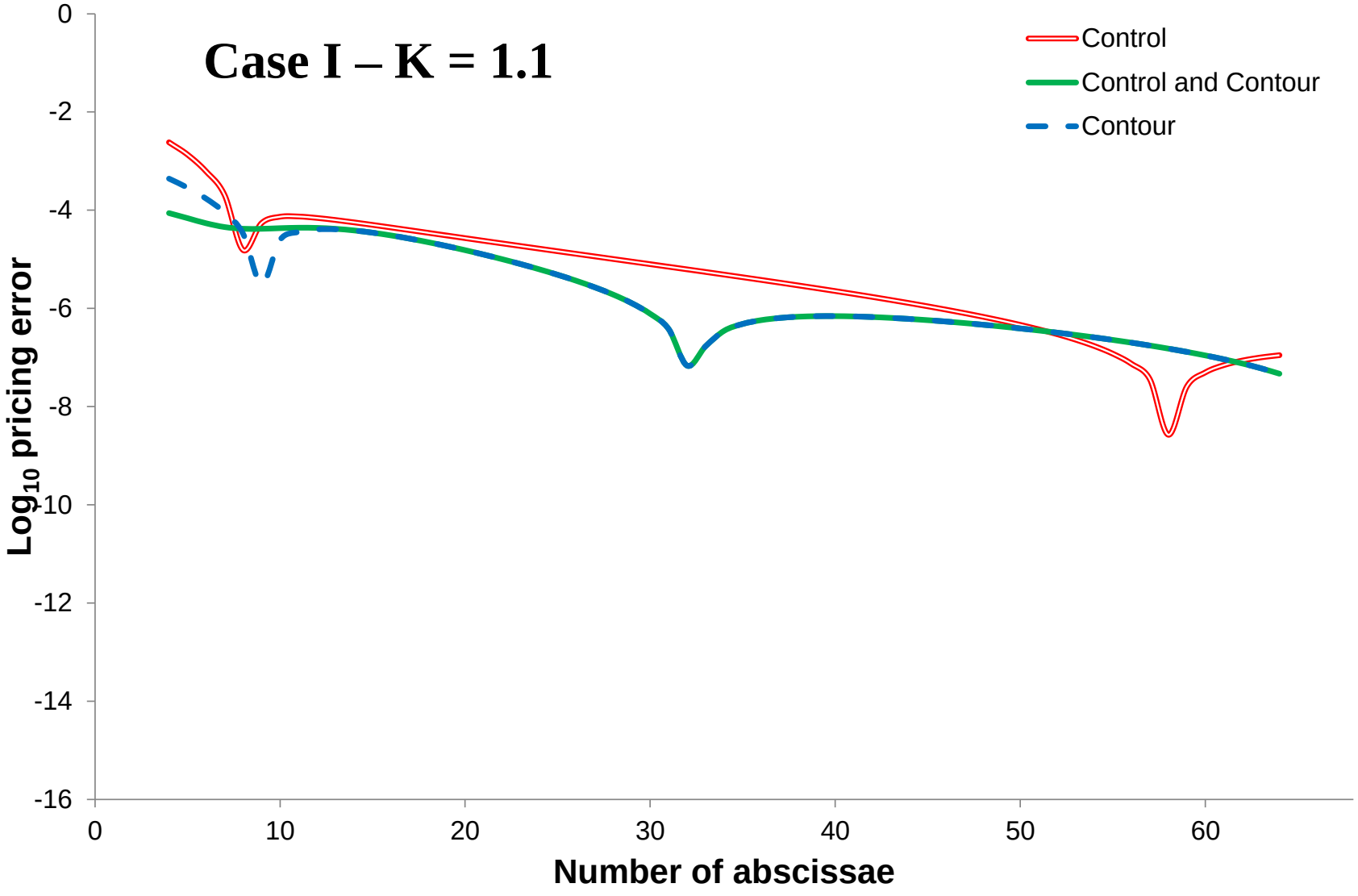
Control and contour (4)



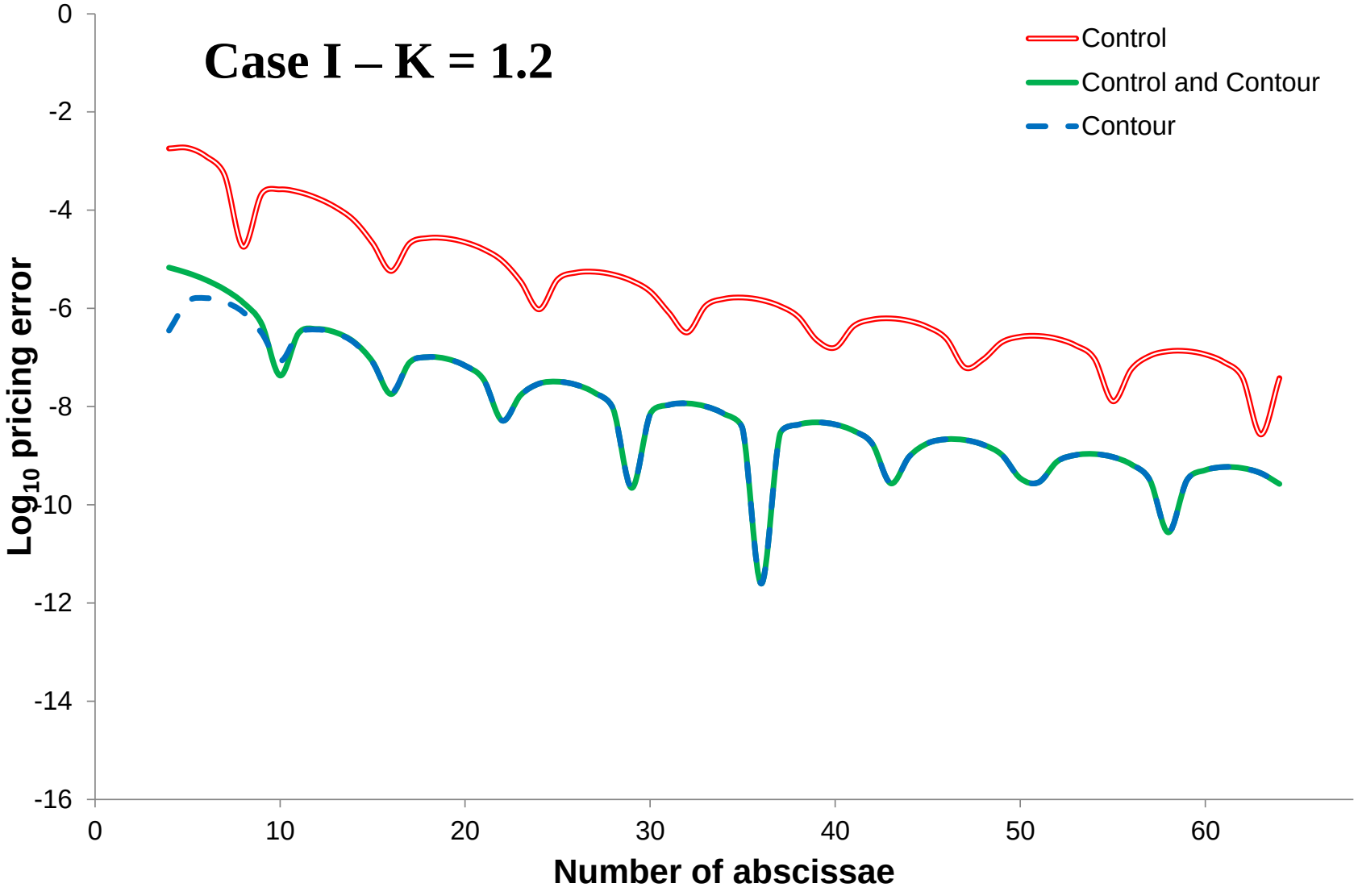
Control and contour (5)



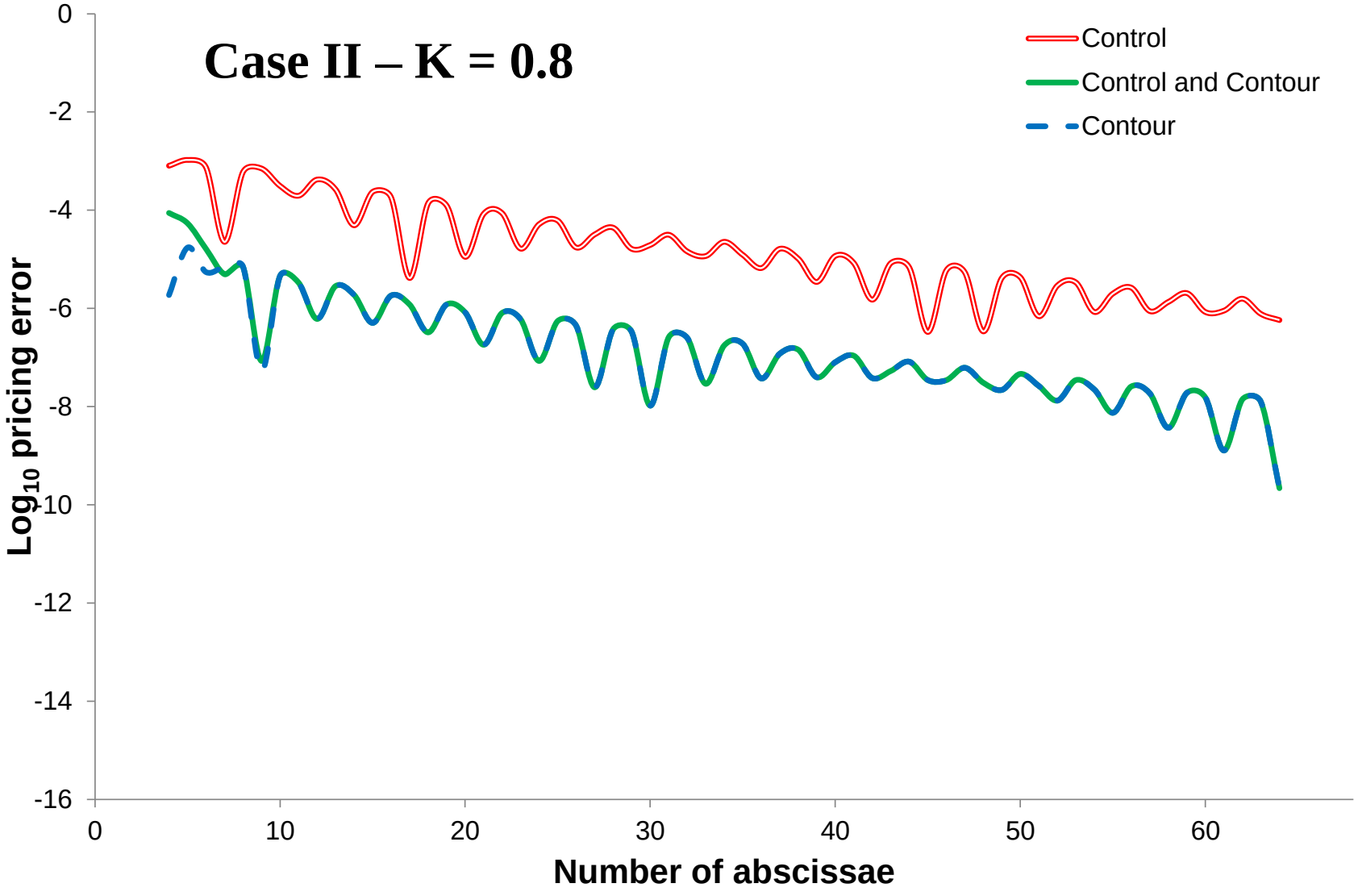
Control and contour (6)



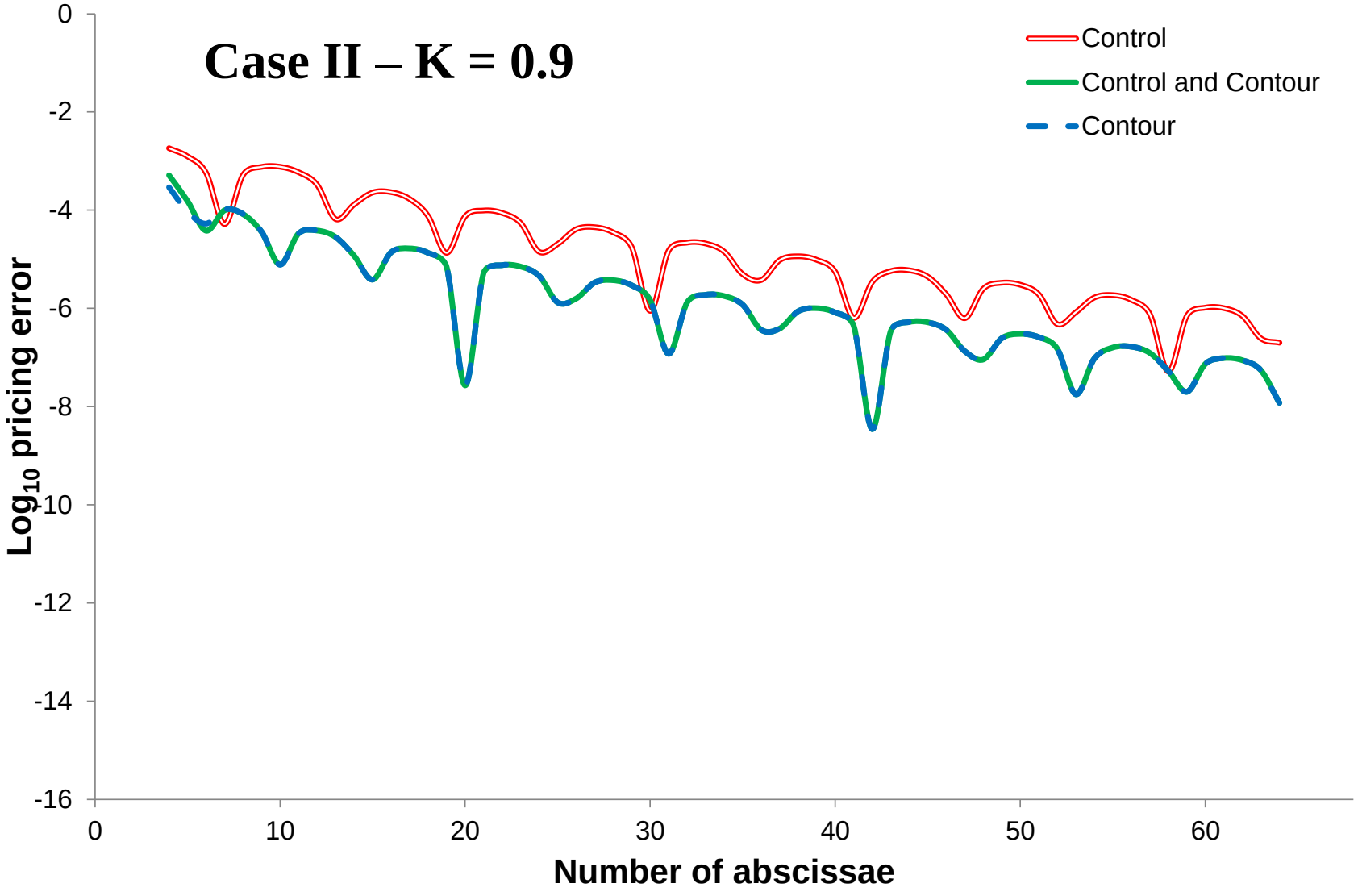
Control and contour (7)



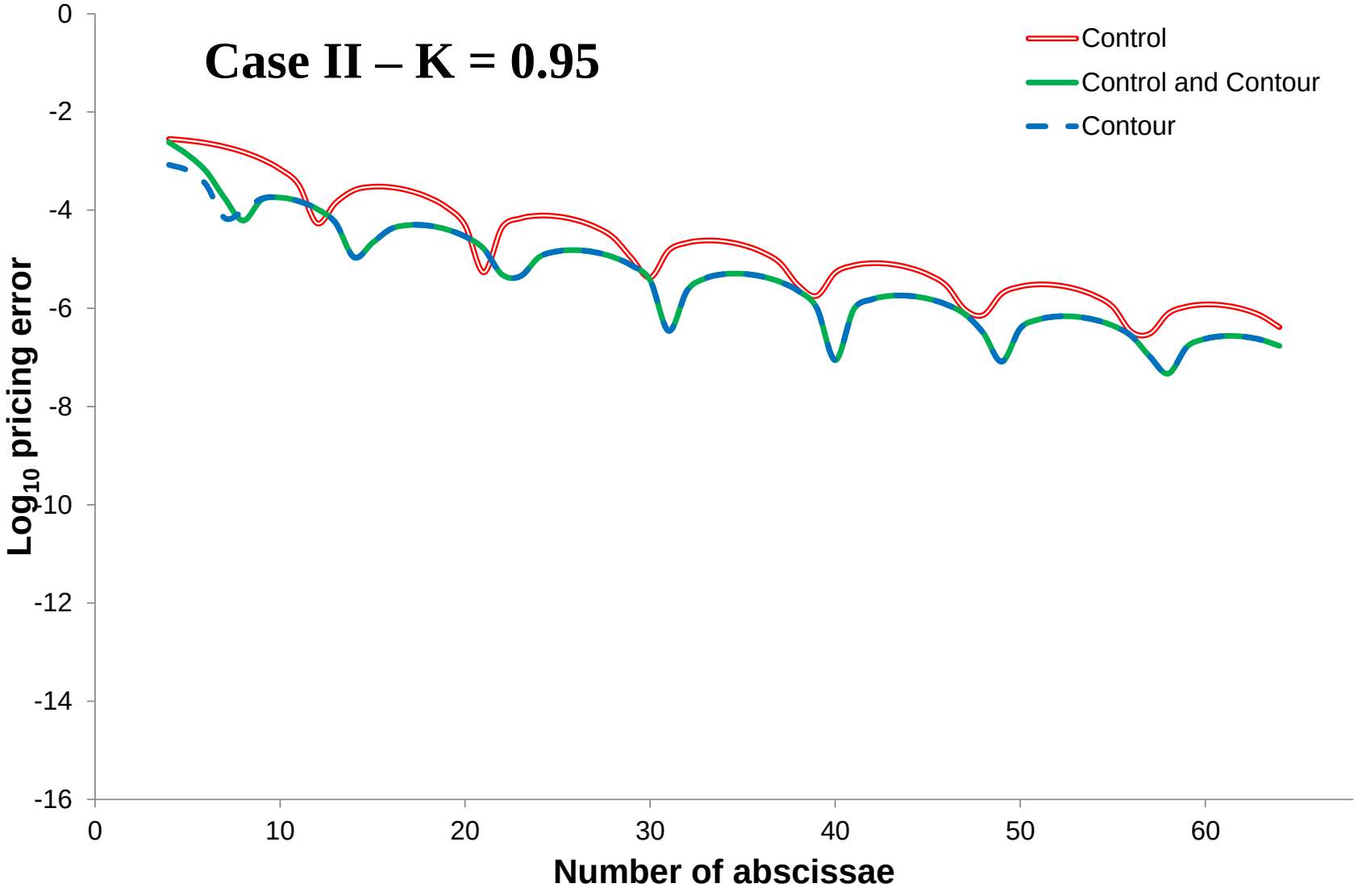
Control and contour (8)



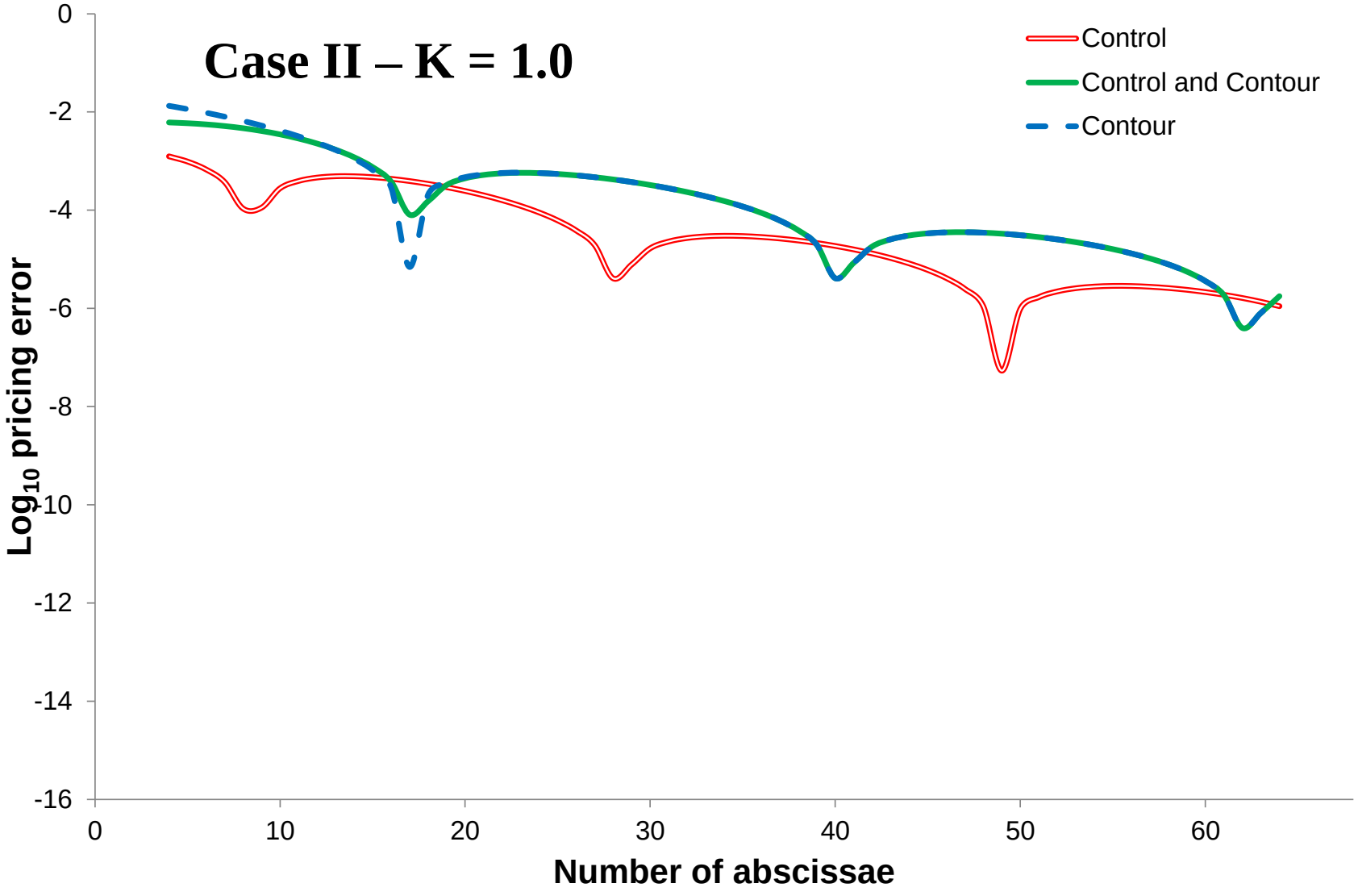
Control and contour (9)



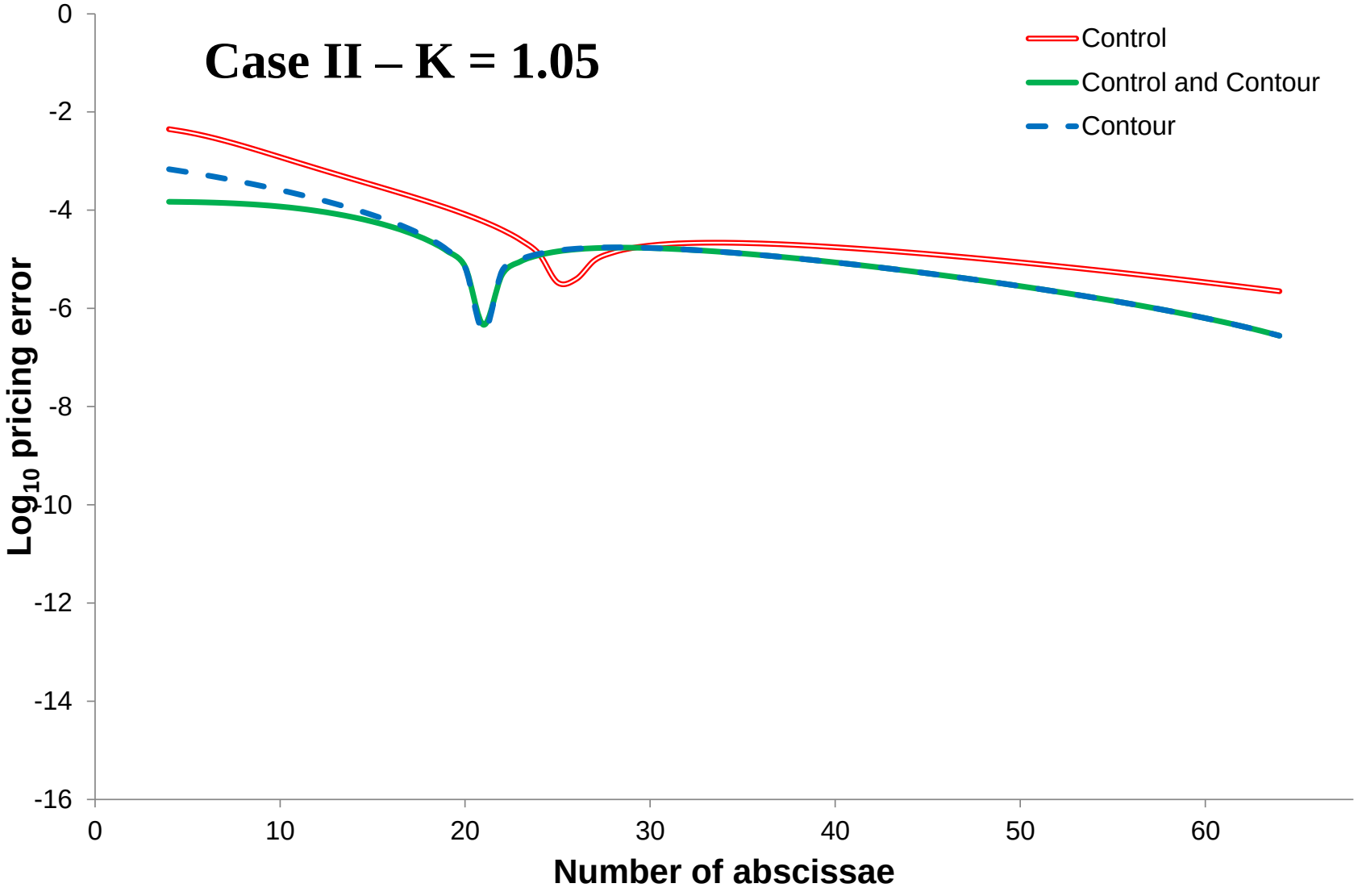
Control and contour (10)



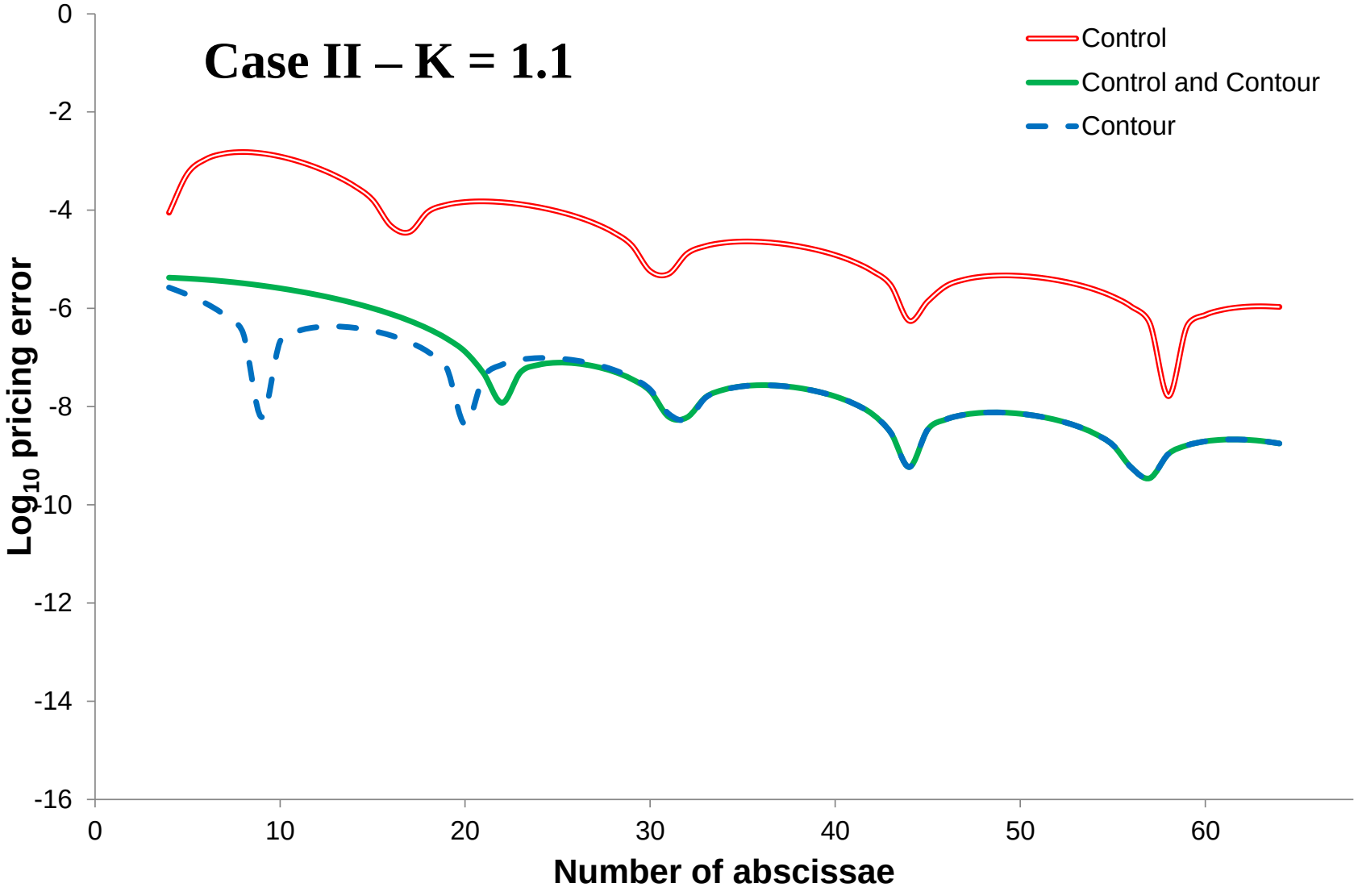
Control and contour (11)



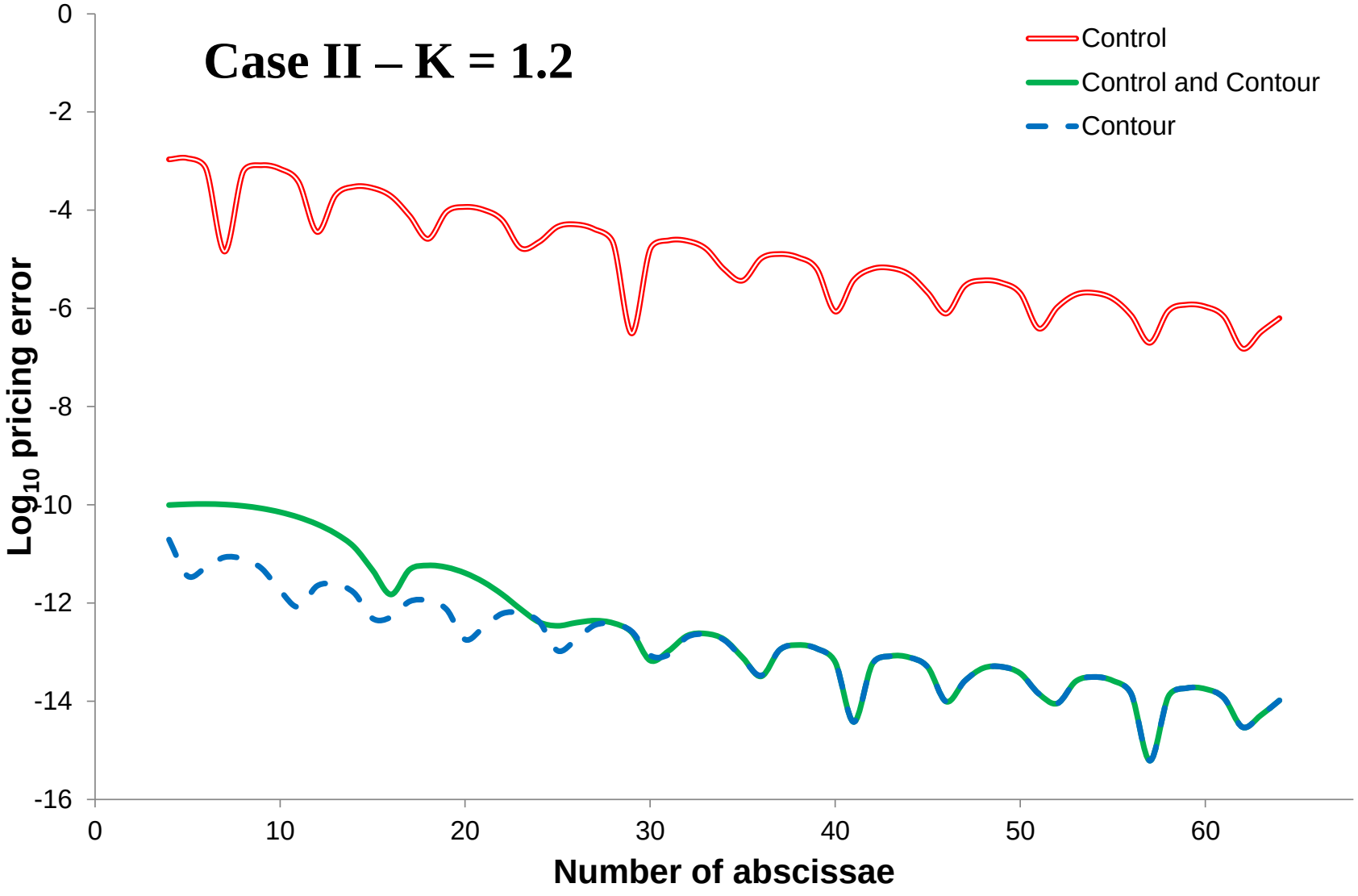
Control and contour (12)



Control and contour (13)



Control and contour (14)



Conclusions

Concluding, we find that:

- For at-the-money options (or around), Joshi and Yang's proposed algorithm has a slight edge to using the optimal contour;
- The optimal contour is the preferred method for strikes that are further away from at-the-money;
- Combining both adds value for a low number of abscissae only (or when other quadrature is used)
- Impact of quadrature and better controls are left for further research